Approximation of general shell problems by flat plate elements (part 2 : addition of drilling degree of freedom)

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Abstract :

In this paper, we analyse an approximation of general thin shell problems where the middle surface is approached by flat triangular facets, whereas the displacement field is approximated by triangles of type (1) for the membrane components and by reduced H.C.T. triangles for the bending component.

In this second part of the paper, we define a sixth degree of freedom : *the rotation around the normal*. This introduces a "small" perturbation but has the advantage to make the implementation easier : indeed, the connection between two adjacent facets is simply realized by imposing the continuity of the displacement and rotation vectors at the vertices of the triangulation.

We prove the "pseudo-convergence" of this method for sufficiently shallow shells ; then we propose a new expression of the bending terms upon each facet for which the approximation method is unconditionally convergent, for arbitrary thin shells.

APPROXIMATION DE PROBLEMES DE COQUES GENERALES PAR DES ELEMENTS DE PLAQUES

(Partie 2 : Addition d'un degré de liberté de rotation)

Résumé :

Dans cet article, nous analysons une approximation de problèmes de coques minces générales pour laquelle la surface moyenne est approchée par un assemblage de facettes planes triangulaires, alors que le champ de déplacement est approché par des triangles de type (1) pour les composantes de membrane et par des triangles H.C.T. réduit pour la composante de flexion.

Dans cette deuxième partie de l'article, on définit un sixième degré de liberté : la rotation autour de la normale. Ceci introduit une "petite" perturbation mais présente l'avantage d'une implémentation facilitée : la connexion entre deux facettes adjacentes est réalisée simplement en imposant la continuité des vecteurs déplacements et rotations aux sommets de la triangulation.

Nous prouvons la "pseudo-convergence" de la méthode pour des coques faiblement courbées ; puis nous proposons une nouvelle expression des termes de flexion sur chaque facette pour laquelle la méthode d'approximation est inconditionnellement convergente, quelque soit la géométrie de la coque mince.

1 INTRODUCTION

For the analysis of general thin shell problems by displacement models, there exists various finite element approach :

- the conforming methods, with the approximation of the geometry in explicit form (CIARLET (1976)), or in implicit form (BERNADOU (1980)) ;
- methods which are *conforming for the displacements* and *nonconforming for the geometry* (see BERNADOU-DUCATEL-TROUVÉ (Part 1)) ;
- nonconforming methods for the displacements (TROUVÉ (to appear)) ;
- completely nonconforming methods, for the displacements and the geometrical approximation (KIKUCHI (1984)).

In the first part of this work, we analyse the approximation of thin shell problems by a CLOUGH-JOHNSON method using *flat plate elements*. The construction of the discrete spaces takes two steps :

(i) *to each facet*, is associated a *conforming plate finite element*, i.e., P_1-triangles for membrane displacements and a reduced H.C.T. triangle for the bending displacement ;

(ii) the definition of *compatibility conditions* that have to be satisfied at each vertex of the triangulation, i.e. the continuity of the displacement vectors and the continuity of the tangential components (taken with respect to the plane tangent to the continuous middle surface) of the rotation vectors. The purpose of these compatibility conditions is : firstly, to enable the definition of a single set of degrees of freedom at each vertex ; secondly, to ensure the *consistency* between the sum of the elementary energies associated to each plane facet, and the energy associated to the continuous surface.

From the *convergence* point of view, we have established :

- the consistency, without restriction, of the membrane strain energies ;
- the consistency of the bending strain energies for a class of *shallow shells* ;
- some lacks of convergence in the case of a non-shallow circular cylinder ;
- the convergence of a new method for arbitrary thin shells, by introducing a perturbation of the bending terms associated to each facet.

Such results appear to be quite new ; and it seems that they do not rely on the particular choice of the shell theory, neither on the finite element type (as far as general shells are concerned).

In this second part of the paper, we study :

(i) the combined effect of a CLOUGH-JOHNSON flat plate elements approximation, on the one hand, and of a *perturbation* associated to the introduction of a *sixth degree of freedom* to each node of the triangulation, on the other hand ; this last degree of freedom would appear as a component of the *rotation around the normal*, i.e. the *drilling rotation* (in this way, let us mention the examples of BATHE-HO (1981), BERGAN-FELIPPA (1985), OLSON-BEARDEN (1979)).

The introduction of this sixth degree of freedom facilitates the implementation of the compatibility conditions : it allows to prescribe the continuity of the displacement and rotation vectors at each node of the triangulation. The present method consists to add a perturbation term to the expression of the energy by affecting to the sixth degree of freedom a *fictitious stiffness coefficient* k. This coefficient k must be taken "sufficiently small" not to affect significantly the response of the system, and "large enough" to avoid the ill-conditionning of the global stiffness matrix. We will prove that the introduction of a sixth degree of freedom does not change fundamentally convergence results, and we determine the bounds for the admissible values of the coefficient k.

(ii) a new plane facet element method, unconditionally convergent for general shells, that only necessitates the knowledge of the euclidean coordinates of each vertex of the triangulation to describe the shell geometry.

Notations and references : In this study, we will use as constant references the notations and results of BERNADOU-DUCATEL-TROUVÉ (Part 1) (*) ; specific formula or theorems of this work will be recorded by adding a "I" symbol.

For instance, the variational formulation of the continuous problem is given in (I, §2), and some results concerning a conforming finite element approximation are presented in (I, §3).

□

(*) ... which summarizes BERNADOU-DUCATEL (Rapports de Recherche INRIA n° 660 et 674) et BERNADOU-DUCATEL-TROUVÉ (Rapport de Recherche INRIA n° 667).

CONTENTS

2 A NONCONFORMING METHOD : ASSEMBLAGE OF FLAT PLATE ELEMENTS

Subsequently, we shall assume the *existence* of a mapping $\vec{\phi}$ such that $\mathcal{S} = \vec{\phi}(\Omega)$, Ω being an open bounded subset of the plane $\mathcal{E}^2$, and that the points $\sigma = \vec{\phi}(\Sigma)$, where Σ denotes the vertices of the triangulation $\mathcal{T}_h$ of Ω, are *known*. As far as the implementation of the present approach is concerned, the *explicit* knowledge of $\vec{\phi}$ is not necessary.

The approximate middle surface $\overline{\mathcal{S}}_h$ being defined in (I, section 4.1), we consider the plate elements of CLOUGH-JOHNSON (1968, 1970) - see the definition of the discrete space $\vec{\vec{X}}_h = \tilde{X}_{h1} \times \tilde{X}_{h1} \times \tilde{X}_{h2}$ in (I, section 4.2) -, and the approximation of a sixth degree of freedom involving a perturbation in the variational formulation of the discrete problem. This approach enables to set very simple compatibility conditions.

2.1. The compatibility conditions

It is essential to introduce compatibility conditions at the nodes of the mesh in order to :

- define at each vertex of $\mathscr{T}_h$ one and only one set of degrees of freedom, say global degrees of freedom ; this point of view was adopted by BERNADOU-DUCATEL-TROUVÉ (Part I), but we consider here another mean of writing such compatibility relations by introducing a sixth degree of freedom at each node ;

- obtain an approximation of the energy which should be consistent with the "continuous" expression of the shell energy.

In this way, we consider two facets k^+ and k^- of the approximate middle surface $\mathscr{S}_h$ which have a common vertex σ (see I, Fig. 4.1.1). This common vertex can be regarded as

(i) *a vertex of the facet* $k^+ = \vec{\phi}_h(K^+)$; thus, any displacement field $\overset{\approx}{\vec{v}}_h \in \overset{\approx}{\vec{X}}_h$ of the surface $\mathscr{S}_h$ has the following components at the point $\sigma^+ = \sigma$ (or at the point Σ^+ of Ω, using $\vec{\phi}_h$) : $\overset{\approx}{\vec{v}}_h(\Sigma^+) = \overset{\approx}{v}_{hi}(\Sigma^+).\vec{a}_h^{\,i}(\Sigma^+)$. To the displacement field $\overset{\approx}{\vec{v}}_h$, we associate the rotation vector $\overset{\approx}{\vec{\omega}}_h$ as follows :

$$\overset{\approx}{\vec{\omega}}_h(\Sigma^+) = \overset{\approx}{\omega}{}_h^i(\Sigma^+).\vec{a}_h^{\,i+} , \tag{2.1.1}$$

where :

$$\overset{\approx}{\omega}{}_h^\lambda(\Sigma^+) = \frac{1}{\sqrt{a_h^+}}\, e^{\lambda\mu}\overset{\approx}{v}_{h3,\mu}(\Sigma^+) , \tag{2.1.2}$$

$\overset{\approx}{\omega}{}_h^3(\Sigma^+)$ is a degree of freedom, independent of $\overset{\approx}{\vec{v}}{}_h^+$ (to be determined) ; (2.1.3)

(ii) *a vertex of the facet* $k^- = \vec{\phi}_h(K^-)$: it suffices to replace the upper script + by - ;

(iii) *a point of the middle surface* $\mathscr{S}$: then, using the notations of (I, section 2.1), any displacement field $\vec{v} \in \vec{V}$ has the components : $\vec{v}(\Sigma) = v_i(\Sigma)\vec{a}^{\,i}(\Sigma)$, at the point $\Sigma = \vec{\phi}^{-1}(\sigma)$; and we can associate to $\vec{v}$ the rotation vector given by (see KOITER (1970)) :

$$\vec{\omega}(\vec{v}) = \omega^i(\vec{v}).\vec{a}_i \text{ , where : } \omega^\lambda(\vec{v}) = \epsilon^{\lambda\mu}(v_{3,\mu}+b_\mu^\nu v_\nu) \text{ , } \omega^3(\vec{v}) = \frac{1}{2}\,\epsilon^{\lambda\mu}v_{\mu,\lambda} \text{ .} \tag{2.1.4}$$

Remark 2.1.1 : Let us already point out that, by construction, the components $\overset{\approx}{\omega}{}_h^\lambda(\Sigma^+)$ are approximations of $\omega^\lambda(\Sigma)$, whereas it is not the case of $\overset{\approx}{\omega}{}_h^3(\Sigma^+)$ which is a degree of freedom *independent* of $\overset{\approx}{\vec{v}}{}_h^+$. This is fundamental if one wants to define compatibility conditions, easy to implement.

□

Therefore, we adopt the following *compatibility conditions* :

(1) *the displacement* $\vec{\vec{v}}_h$ *is continuous at the nodes* σ *of the surface* $\mathcal{S}$, *or equivalently, at the vertices* Σ *of the triangulation* $\mathcal{T}_h$, *i.e.*

$$\vec{\vec{v}}_h(\Sigma^+) = \vec{\vec{v}}_h(\Sigma^-) \ , \quad \forall\Sigma \text{ vertex of } \mathcal{T}_h \ ; \tag{2.1.5}$$

(2) *the rotation* $\vec{\vec{\omega}}_h$ *is continuous at the nodes* σ *of the surface* $\mathcal{S}$, *or equivalently, at the vertices* Σ *of the triangulation* $\mathcal{T}_h$, *i.e.*

$$\vec{\vec{\omega}}_h(\Sigma^+) = \vec{\vec{\omega}}_h(\Sigma^-) \ , \quad \forall\Sigma \text{ vertex of } \mathcal{T}_h \ . \tag{2.1.6}$$

It remains to verify that the compatibility conditions (2.1.5) and (2.1.6) allow to correctly define :

- a finite element space $\vec{Y}_h$ (then $\vec{W}_h$ by taking into account the boundary conditions) ;

-a discrete problem.

2.2. The discrete spaces $\vec{Y}_h$ *and* $\vec{W}_h$

A convenient way to introduce the sixth degree of freedom at each vertex of the triangle K of $\mathcal{T}_h$ consists in the following definition of the discrete space $\vec{Y}_h$:

$$\left.\begin{aligned}\vec{Y}_h = \{(\vec{\vec{v}}_h,\tilde{\tilde{\omega}}_h^3) \in \vec{\vec{X}}_h \times \tilde{\tilde{X}}_{h1} \ ; \ (\vec{\vec{v}}_h,\tilde{\tilde{\omega}}_h^3) \text{ satisfies the compatibility} \\ \text{conditions (2.1.5) and (2.1.6)}\} \ .\end{aligned}\right\} \tag{2.2.1}$$

By noticing that the previous compatibility conditions enable to associate to each function $(\vec{v}_h,\tilde{\omega}_h^3) \in \vec{Y}_h$ a single set of values $\{\vec{v}_h(\Sigma),\vec{\omega}_h(\Sigma)\}$ at each vertex Σ of $\mathcal{T}_h$, we can show that : $\dim \vec{Y}_h = 6N_h$, where N_h is the number of vertices of the triangulation $\mathcal{T}_h$. A practical manner to define the unique set of values $\{\vec{v}_h(\Sigma),\vec{\omega}_h(\Sigma),\ \Sigma \text{ vertex of } \mathcal{T}_h\}$ is to express the components of the vectors $\vec{v}_h(\Sigma),\vec{\omega}_h(\Sigma)$ on the orthonormal reference system $(0,\vec{e}_1,\vec{e}_2,\vec{e}_3)$.

Remark 2.2.1 : The definitions of the spaces $\tilde{\tilde{X}}_{h1}$ and $\tilde{\tilde{X}}_{h2}$ (see (I, section 4.2) and (I, (3.1.1), (3.1.2)) imply that : $\dim \vec{\vec{X}}_h = 15M_h$, where M_h is the number of triangles of $\mathcal{T}_h$; then we get $\dim(\vec{\vec{X}}_h \times \tilde{\tilde{X}}_{h1}) = 18M_h$. Such a discrete space $(\vec{\vec{X}}_h \times \tilde{\tilde{X}}_{h1})$ involves the following *local* degrees of freedom, associated with each triangle K of $\mathcal{T}_h$ with vertices Σ_1, Σ_2, Σ_3 :

$$[DDL_K(\vec{\tilde{\tilde{v}}}_h,\tilde{\tilde{\omega}}^3_h)] = [\tilde{\tilde{v}}_{h1}(\Sigma_1) \; ; \; \tilde{\tilde{v}}_{h1}(\Sigma_2) \; ; \; \tilde{\tilde{v}}_{h1}(\Sigma_3) \; ; \; \tilde{\tilde{v}}_{h2}(\Sigma_1) \; ; \; \tilde{\tilde{v}}_{h2}(\Sigma_2) \; ; \; \tilde{\tilde{v}}_{h2}(\Sigma_3) \; ;$$
$$\tilde{\tilde{v}}_{h3}(\Sigma_1) \; ; \; \tilde{\tilde{v}}_{h3}(\Sigma_2) \; ; \; \tilde{\tilde{v}}_{h3}(\Sigma_3) \; ; \; \tilde{\tilde{v}}_{h3,1}(\Sigma_1) \; ; \; \tilde{\tilde{v}}_{h3,2}(\Sigma_1) \; ;$$
$$\tilde{\tilde{v}}_{h3,1}(\Sigma_2) \; ; \; \tilde{\tilde{v}}_{h3,2}(\Sigma_2) \; ; \; \tilde{\tilde{v}}_{h3,1}(\Sigma_3) \; ; \; \tilde{\tilde{v}}_{h3,2}(\Sigma_3) \; ; \; \tilde{\tilde{\omega}}^3_h(\Sigma_1) \; ;$$
$$\tilde{\tilde{\omega}}^3_h(\Sigma_2) \; ; \; \tilde{\tilde{\omega}}^3_h(\Sigma_3)] \; ;$$

and the corresponding elementary stiffness matrix is block diagonal, i.e.

$$[RL_K] = \left[\begin{array}{c|c|c} R_{Km} & 0 & 0 \\ \hline 0 & R_{Kb} & 0 \\ \hline 0 & 0 & R_{Kd} \end{array}\right] ,$$

where R_{Km} is a 6×6 membrane stiffness matrix, R_{Kb} a 9×9 bending stiffness matrix, and R_{Kd} a 3×3 drilling stiffness matrix. In such a manner the functions $(\vec{\tilde{\tilde{v}}}_h,\tilde{\tilde{\omega}}^3_h) \in \vec{\tilde{\tilde{X}}}_h \times \tilde{\tilde{X}}_{h1}$ are determined *triangle by triangle*. Now, by using the compatibility conditions (2.1.5) and (2.1.6), we define a new set of *global* degrees of freedom, associated with each vertex Σ of $\mathcal{T}_h$, i.e.

$$\tilde{v}_{hi}(\Sigma) = \tilde{\tilde{v}}_{hk}(\Sigma).(\vec{a}^k_h.\vec{e}_i) \quad , \quad 1\le i\le 3 \; ,$$

$$\tilde{\omega}^j_h(\Sigma) = \frac{1}{\sqrt{a_h}} \, e^{\lambda\mu}\tilde{\tilde{v}}_{h3,\mu}(\Sigma) \; (\vec{a}_{h\lambda}.\vec{e}^j) + \tilde{\tilde{\omega}}^3_h(\Sigma) \; (\vec{a}_{h3}.\vec{e}^j) \quad , \quad 1\le j\le 3 \; ,$$

where $\vec{e}^j = \vec{e}_j$, $1\le j\le 3$. Conversely, the relations :

$$\tilde{\tilde{v}}_{hi}(\Sigma) = \tilde{v}_{hk}(\Sigma) \; (\vec{e}^k.\vec{a}_{hi}) \quad , \quad 1\le i\le 3 \; ,$$

$$\tilde{\tilde{v}}_{h3,\mu}(\Sigma) = \sqrt{a_h} \; e_{\lambda\mu}\tilde{\omega}^j_h(\Sigma) \; (\vec{e}_j.\vec{a}^\lambda_h) \quad , \quad \mu = 1,2 \; ,$$

$$\tilde{\tilde{\omega}}^3_h(\Sigma) = \tilde{\omega}^j_h(\Sigma) \; (\vec{e}_j.\vec{a}^3_h) \quad ,$$

enable to define a 18×18 matrix $[P_K]$ such that :

$$[DDL_K(\vec{\tilde{\tilde{v}}}_h,\tilde{\tilde{\omega}}^3_h)] = [DDG_K(\vec{\tilde{v}}_h,\vec{\tilde{\omega}}_h)].[P_K] \quad ,$$

where

$$[DDG_K(\vec{\tilde{v}}_h,\vec{\tilde{\omega}}_h)] = [\tilde{v}_{h1}(\Sigma_i) \; ; \; \tilde{v}_{h2}(\Sigma_i) \; ; \; \tilde{v}_{h3}(\Sigma_i) \; ; \; \tilde{\omega}^1_h(\Sigma_i) \; ; \; \tilde{\omega}^2_h(\Sigma_i) \; ; \; \tilde{\omega}^3_h(\Sigma_i))_{i=1,2,3}] \; .$$

In such a way, we are able to evaluate the elementary stiffness matrix in its global form (which is not block diagonal any more), i.e.

$$[RG_K] = [P_K].[RL_K].{}^t[P_K] .$$

Then, it suffices to assemble the elementary matrices RG_K, for any triangle K of $\mathscr{T}_h$, in order to derive the global stiffness matrix.

□

Next, we equip the discrete space $\vec{\vec{Y}}_h$ with the norm (compare with (I, (4.2.15)))

$$(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{Y}}_h \to \sum_{K\in\mathscr{T}_h} [\sum_{\alpha=1}^{2} \|\tilde{v}_{h\alpha}\|_{1,K}^2 + \|\tilde{v}_{h3}\|_{2,K}^2 + \|\tilde{\omega}_h^3\|_{0,K}^2]^{1/2} . \tag{2.2.2}$$

Existence of a bijection between the discrete spaces $\vec{\vec{Y}}_h$ *and* $\vec{Y}_h = \vec{X}_h \times X_{h1}$

As the functions $(\vec{\vec{v}}_h,\tilde{\omega}_h^3)$ of the space $\vec{\vec{Y}}_h$ are completely determined by the data of $6N_h$ degrees of freedom, we will associate to this set of values the set $\{\vec{v}_h(\Sigma),\vec{\omega}_h(\Sigma),\ \Sigma$ vertex of $\mathscr{T}_h\}$, by means of a simple *extension* of the compatibility conditions, i.e.

$$\vec{v}_h(\Sigma) = \vec{\vec{v}}_h(\Sigma) \text{ and } \vec{\omega}_h(\Sigma) = \vec{\vec{\omega}}_h(\Sigma) \quad , \quad \forall\Sigma \text{ vertex of } \mathscr{T}_h . \tag{2.2.3}$$

The set of these new values is associated to the continuous middle surface $\mathscr{S}$, and we show in Theorem 2.2.1 that it defines one and only one function $(\vec{v}_h,\omega_h^3) \in (\vec{X}_h,X_{h1})$; therefore, we introduce the discrete space $\vec{Y}_h$:

$$\vec{Y}_h = \{(\vec{v}_h,\omega_h^3) \in \vec{X}_h \times X_{h1}\} \subset (H^1(\Omega))^2 \times H^2(\Omega) \times H^1(\Omega) \quad , \tag{2.2.4}$$

(equipped with the usual product norm).

Theorem 2.2.1 : The compatibility relations (2.2.3) *define a bijection* $F_h : \vec{\vec{Y}}_h \to \vec{Y}_h$, *that associates to each function* $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{Y}}_h$ *one and only one function* $(\vec{v}_h,\omega_h^3) \in \vec{Y}_h$.

Proof : (i) $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{Y}}_h \to (\vec{v}_h,\omega_h^3) \in \vec{Y}_h$

It is a simple consequence of the definitions (2.1.4), (2.2.1) and of the relations (2.2.3) : we find the expressions

$$
\left.
\begin{aligned}
& v_{hi}(\Sigma) = \overset{\rightrightarrows}{v}_h(\Sigma).\vec{a}_i(\Sigma) \quad , \quad 1 \le i \le 3 \ , \\
& v_{h3,\mu}(\Sigma) = \sqrt{a(\Sigma)}\ e_{\lambda\mu}[\overset{\rightrightarrows}{\omega}_h(\Sigma).\vec{a}^{\lambda}(\Sigma)] - b^{\lambda}_{\mu}(\Sigma)\ \overset{\rightrightarrows}{v}_h(\Sigma).\vec{a}_{\lambda}(\Sigma) \quad , \\
& \omega^3_h(\Sigma) = \overset{\rightrightarrows}{\omega}_h(\Sigma).\vec{a}^3(\Sigma) \quad ,
\end{aligned}
\right\} \tag{2.2.5}
$$

which completely determine the function $(\vec{v}_h, \omega^3_h) \in \vec{Y}_h$ - Moreover, the relations (2.2.5) clearly define an injection F_h .

(ii) $(\vec{v}_h, \omega^3_h) \in \vec{Y}_h \to (\overset{\rightrightarrows}{v}_h, \tilde{\omega}^3_h) \in \overset{\rightrightarrows}{Y}_h$

Conversely, from the relations (2.2.3) and the definitions (2.1.1) (2.1.2) (2.1.3), we obtain for each triangle K^+ of $\mathscr{T}_h$:

$$
\overset{\approx}{v}_{hi}(\Sigma^+) = v_{hj}(\Sigma)(\vec{a}^j(\Sigma).\vec{a}^{+}_{hi}) \quad , \quad 1 \le i \le 3 \quad , \tag{2.2.6}
$$

$$
\left.
\begin{aligned}
\overset{\approx}{v}_{h3,\mu}(\Sigma^+) = {} & \sqrt{\frac{a^+_h}{a(\Sigma)}}\ e_{\lambda\mu} e^{\kappa\nu}[v_{h3,\nu}(\Sigma) + b^{\epsilon}_{\nu}(\Sigma) v_{h\epsilon}(\Sigma)](\vec{a}_{\kappa}(\Sigma).\vec{a}^{\lambda+}_h) + \\
& + \sqrt{a^+_h}\ e_{\lambda\mu}\omega^3_h(\Sigma)(\vec{a}_3(\Sigma).\vec{a}^{\lambda+}_h) \quad ,
\end{aligned}
\right\} \tag{2.2.7}
$$

$$
\left.
\begin{aligned}
\overset{\approx}{\omega}{}^3_h(\Sigma^+) = {} & \frac{1}{\sqrt{a(\Sigma)}}\ e^{\kappa\nu}[v_{h3,\nu}(\Sigma) + b^{\epsilon}_{\nu}(\Sigma) v_{h\epsilon}(\Sigma)](\vec{a}_{\kappa}(\Sigma).\vec{a}^{3+}_h) + \\
& + \omega^3_h(\Sigma)(\vec{a}_3(\Sigma)\vec{a}^{3+}_h) \quad ;
\end{aligned}
\right\} \tag{2.2.8}
$$

which determine a unique element $(\overset{\approx}{\overset{\rightrightarrows}{v}}_h, \overset{\approx}{\omega}{}^3_h) \in \overset{\approx}{\overset{\rightrightarrows}{X}}_h \times \overset{\approx}{X}_{h1}$ - And, we easily check that this element $(\overset{\approx}{\overset{\rightrightarrows}{v}}_h, \overset{\approx}{\omega}{}^3_h)$ satisfies the compatibility conditions (2.1.5) and (2.1.6), i.e. $(\overset{\rightrightarrows}{v}_h, \overset{\approx}{\omega}{}^3_h) \in \overset{\rightrightarrows}{Y}_h$.

(iii) Finally, by combining the relations (2.2.5) and (2.2.6) to (2.2.8), it is seen that the application defined in step (ii) is the inverse F_h^{-1} of the injection F_h defined in step (i).

□

The discrete spaces $\overset{\rightrightarrows}{W}_h$ *and* $\vec{W}_h$

In order to take into account the boundary conditions of the *clamped type* on the edge $\vec{\phi}(\Gamma_o)$, i.e. $\vec{v}_{h|\Gamma_o} = \vec{0}$, $\partial_{\nu} v_{h3|\Gamma_o} = 0$, we introduce the discrete space

$$
\overset{\rightrightarrows}{W}_h = \{(\overset{\rightrightarrows}{v}_h, \tilde{\omega}^3_h) \in \overset{\rightrightarrows}{Y}_h \ ; \ \overset{\rightrightarrows}{v}_h(\Sigma) = \vec{0} \text{ and } \overset{\rightrightarrows}{\omega}_h(\Sigma) = \vec{0} \quad , \quad \forall \Sigma \in \Gamma_o\} \ . \tag{2.2.9}
$$

Indeed, such a definition leads, in relation (2.2.5), to the recovering of the conditions : $\vec{v}_h(\Sigma) = \vec{0}$; $v_{h3,\mu}(\Sigma) = 0$, $\mu = 1,2$, and $\omega_h^3 = 0$, $\forall\Sigma \in \Gamma_o$. Conversely, such boundary conditions give, in the relations (2.2.6) to (2.2.8), the boundary conditions of the definition (2.2.9). Thus, the bijection F_h, defined in Theorem 2.2.1, enables to introduce the discrete space $\vec{W}_h = F_h(\vec{\vec{W}}_h)$, i.e.

$$\vec{W}_h = \{(\vec{v}_h,\omega_h^3) \in \vec{Y}_h \ ; \ \vec{v}_{h|\Gamma_o} = \vec{0} \text{ and } \vec{\omega}_{h|\Gamma_o} = \vec{0}\} \quad . \tag{2.2.10}$$

Remark 2.2.2 : In BERNADOU-DUCATEL-TROUVÉ (Part I), it has been pointed out that the boundary conditions along $\vec{\phi}_h(\Gamma_o)$ were not of the clamped type. In a similar way, the conditions : $\vec{v}_h(\Sigma) = \vec{0}$ and $v_{h3,\alpha}(\Sigma) = 0$, $\alpha = 1,2$, $\forall\Sigma \in \Gamma_o$, would only lead here to :

$$\left.\begin{aligned} &\vec{\vec{\vec{v}}}_h(\Sigma^+) = \vec{0} \ , \\ &\tilde{\tilde{v}}_{h3,\mu}(\Sigma^+) = e_{\lambda\mu}\,\frac{\vec{a}_3(\Sigma).\vec{a}_h^{\lambda+}}{\vec{a}_3(\Sigma).\vec{a}_h^{3+}}\,\sqrt{a_h^+}\,\tilde{\tilde{\omega}}_h^3(\Sigma^+) \quad , \ \mu = 1,2 \ , \end{aligned}\right\} \forall\Sigma^+ \in \Gamma_o \quad ,$$

assuming that : $\vec{a}_3(\Sigma).\vec{a}_h^{3+} \neq 0$ (which is easily obtained for h sufficiently small). In general, these conditions involve : $\tilde{\tilde{v}}_{h3|\Gamma_o} \neq 0$ and $\partial_\nu\tilde{\tilde{v}}_{h3|\Gamma_o} \neq 0$. At least, by noticing that we have : $\vec{a}_3(\Sigma).\vec{a}_h^{i+} = \delta_3^i + 0(h)$ (see (I, (5.2.15), (5.2.33))), and by conjecturing that ω_h^3 admits an 0(h)-estimate, we would get : $\tilde{\tilde{v}}_{h3,\mu}(\Sigma^+) = 0(h)$ and $\tilde{\tilde{\omega}}_h^3(\Sigma^+) = 0(h)$, $\forall\Sigma^+ \in \Gamma_o$.

From the implementation point of view, it appears advantageous to introduce a sixth condition, i.e. : $\tilde{\tilde{\omega}}_h^3(\Sigma^+) = 0$, $\forall\Sigma^+ \in \Gamma_o$, so that boundary conditions reduce to : $\vec{\vec{\vec{v}}}_h(\Sigma^+) = \vec{0}$ and $\vec{\vec{\vec{\omega}}}_h(\Sigma^+) = \vec{0}$, $\forall\Sigma^+ \in \Gamma_o$. In particular, these conditions are sufficient to ensure *boundary conditions of the clamped type* for the discrete problems respectively associated to the *faceted middle surface* and the *continuous middle surface*. From a mathematical point of view, the introduction of this sixth boundary condition : $\tilde{\tilde{\omega}}^3_{h|\Gamma_o} = 0$ (resp. $\omega^3_{h|\Gamma_o} = 0$), conserves a sense when h approaches zero, as we have : $\tilde{\tilde{X}}_{h1} \subset \prod_{K\in\mathscr{T}_h} H^1(K)$ (resp. $X_{h1} \subset H^1(\Omega)$). Moreover, let us remark that the usual conforming discrete problem (see (I, §3)) admits a solution $(\vec{u}_h,\eta_h^3)$ that satisfies implicitly $\eta^3_{h|\Gamma_o} = 0$ (in fact $\eta_h^3 = 0$ on $\bar{\Omega}$) .

□

2.3. *The discrete problem associated to the faceted surface*

Let us consider a displacement field $(\vec{\vec{v}}_h,\bar{\omega}_h^3) \in \vec{\vec{Y}}_h$ of the faceted surface $\mathscr{S}_h$.

We can define, triangle by triangle, the components of the corresponding strain tensor $(\tilde{\gamma}_{h\alpha\beta})$ and the components of the change of curvature tensor $(\tilde{\rho}_{h\alpha\beta})$, i.e.

$$\left.\begin{aligned}\tilde{\gamma}_{h\alpha\beta}(\vec{\vec{v}}_h) &= \frac{1}{2}\,(\tilde{v}_{h\beta,\alpha}+\tilde{v}_{h\alpha,\beta}) \quad ,\\ \tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h) &= \tilde{v}_{h3,\alpha\beta} \quad .\end{aligned}\right\} \tag{2.3.1}$$

Then, the elementary bilinear form associated to the triangle $K \in \mathcal{T}_h$ is given by :

$$\tilde{A}_{Kh}[(\vec{\vec{u}}_h,\tilde{\eta}^3_h),(\vec{\vec{v}}_h,\tilde{\omega}^3_h)] = \tilde{a}_{Kh}(\vec{\vec{u}}_h,\vec{\vec{v}}_h) + k(\tilde{\eta}^3_h,\tilde{\omega}^3_h)_{L^2(K)} \quad , \tag{2.3.2}$$

where we have denoted for any $(\vec{\vec{u}}_h,\tilde{\eta}^3_h),(\vec{\vec{v}}_h,\tilde{\omega}^3_h) \in \vec{\vec{Y}}_h$

$$\left.\begin{aligned}\tilde{a}_{Kh}(\vec{\vec{u}}_h,\vec{\vec{v}}_h) = \int_K \frac{Ee}{1-\nu^2}\,\{ &(1-\nu)\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\gamma}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h) +\\ &+ \frac{e^2}{12}\,[(1-\nu)\tilde{\rho}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\rho}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h)]\}\,\sqrt{a_h}\,d\xi^1 d\xi^2 \ ,\end{aligned}\right\} \tag{2.3.3}$$

$k > 0$ constant, independent of K ,

$$(\tilde{\eta}^3_h,\tilde{\omega}^3_h)_{L^2(K)} = \int_K \tilde{\eta}^3_h\tilde{\omega}^3_h\,\sqrt{a_h}\,d\xi^1 d\xi^2 \quad . \tag{2.3.4}$$

Similarly, the elementary linear form is given by :

$$\tilde{f}_{Kh}(\vec{\vec{v}}_h) = \int_K \vec{p}\cdot\vec{\vec{v}}_h\,\sqrt{a_h}\,d\xi^1 d\xi^2 \quad , \quad \forall\vec{\vec{v}}_h \in \vec{\vec{Y}}_h \quad . \tag{2.3.5}$$

Thus, the variational formulation of the *discrete problem associated to the faceted surface* can be stated as :

Problem 2.3.1 : find $(\vec{\vec{u}}_h,\tilde{\eta}^3_h) \in \vec{\vec{W}}_h$ *such that*

$$\sum_{K\in\mathcal{T}_h} \tilde{A}_{Kh}[(\vec{\vec{u}}_h,\tilde{\eta}^3_h),(\vec{\vec{v}}_h,\tilde{\omega}^3_h)] = \sum_{K\in\mathcal{T}_h} \tilde{f}_{Kh}(\vec{\vec{v}}_h) \quad , \quad \forall(\vec{\vec{v}}_h,\tilde{\omega}^3_h) \in \vec{\vec{W}}_h \quad ,$$

with the definitions (2.3.2) *to* (2.3.5).

□

In order to pursue the mathematical analysis of this problem, it is convenient to associate a *new discrete problem formulated on the continuous middle surface* to the previous one. By using the bijection F_h defined in theorem 2.2.1, the discrete problem 2.3.1 may be rewritten in the form of a discrete problem defined over the space $\vec{W}_h$, i.e.

Problem 2.3.2 : find $(\vec{u}^*_h, \eta^3_h) \in \vec{W}_h$ *such that*

$$A_h[(\vec{u}^*_h,\eta^3_h),(\vec{v}_h,\omega^3_h)] = G_h[(\vec{v}_h,\omega^3_h)] \quad , \quad \forall(\vec{v}_h,\omega^3_h) \in \vec{W}_h \quad ,$$

with the following correspondences for any $(\vec{v}_h,\omega^3_h) = F_h(\vec{\vec{v}}_h,\tilde{\omega}^3_h)$, *and any* $(\vec{w}_h,\theta^3_h) = F_h(\vec{\vec{w}}_h,\tilde{\theta}^3_h)$:

$$\left.\begin{aligned} a_{Kh}[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)] &= \tilde{a}_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h) \quad , \\ b_{Kh}[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)] &= (\tilde{\omega}^3_h,\tilde{\theta}^3_h)_{L^2(K)} \quad , \\ f_{Kh}(\vec{v}_h,\omega^3_h) &= \tilde{f}_{Kh}(\vec{\vec{v}}_h) \quad , \end{aligned}\right\} \tag{2.3.6}$$

and where :

$$A_h[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)] = \sum_{K\in\mathcal{T}_h} (a_{Kh}(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)] + k\, b_{Kh}[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)]) \tag{2.3.7}$$

$$G_h[(\vec{v}_h,\omega^3_h)] = \sum_{K\in\mathcal{T}_h} f_{Kh}(\vec{v}_h,\omega^3_h) \quad . \tag{2.3.8}$$

□

Moreover, it is clear that, if the solution $(\vec{\vec{u}}_h,\tilde{\eta}^3_h) \in \vec{\vec{W}}_h$ of the discrete problem 2.3.1 exists, then we have : $(\vec{u}^*_h,\eta^3_h) = F_h(\vec{\vec{u}}_h,\vec{\eta}^3_h)$, with similar expressions as (2.2.5) (this is a consequence of the fact that F_h is a bijection between the discrete spaces $\vec{\vec{W}}_h$ and $\vec{W}_h$).

Let us add that, due to the contribution of ω^3_h in the linear form $G_h[.]$, one should expect that $\eta^3_h \neq 0$ on Ω ; at best we can conjecture that this nonconforming method states an estimate like $\|\eta^3_h\|_{0,\Omega} = O(h)$.

3 ERROR ESTIMATES ; THE CASE OF QUASI-SHALLOW SHELLS

In this paragraph, we establish sufficient conditions on the stiffness coefficient k that ensures the existence and the uniqueness of the solution $(\vec{u}^*_h,\eta^3_h)$ of the discrete problem 2.3.2 ; and we prove the following estimate

$$(\|\vec{u}-\vec{u}^*_h\|^2_{\vec{V}} + \|\eta^3_h\|^2_{0,\Omega})^{1/2} \leq C(h+\epsilon)\ \|\vec{u}\|_{\vec{V}}$$

valid for a class of shallow shells $\mathcal{S}_\epsilon$ such that : $|b^\beta_\alpha| \le \epsilon$, $|b^\beta_{\alpha|\lambda}| \le \epsilon$, where ϵ is a "small" geometrical parameter.

3.1. *Abstract error estimate*

In order to derive error estimates directly between the solutions of the continuous and discrete problems, we give an "abstract" error estimate :

Theorem 3.1.1 : Let us consider a family of discrete problems 2.3.2 *for which there exists a constant* $\beta > 0$ *, independent of* h *, such that*

$$\beta(\|\vec{v}_h\|^2_{\vec{V}} + \|\omega^3_h\|^2_{0,\Omega}) \le A_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \quad , \quad \forall(\vec{v}_h,\omega^3_h) \in \vec{W}_h \quad . \tag{3.1.1}$$

Then, there exists a constant C > 0 *, independent of* h *, such that :*

$$\left.\begin{aligned}
&(\|\vec{u}-\vec{u}^*_h\|^2_{\vec{V}} + \|\eta^3_h\|^2_{0,\Omega})^{1/2} \le C\Big(\inf_{(\vec{v}_h,\omega^3_h)\in\vec{W}_h} [(\|\vec{u}-\vec{v}_h\|^2_{\vec{V}} + \|\omega^3_h\|^2_{0,\Omega})^{1/2} + \\
&+ \sup_{(\vec{w}_h,\theta^3_h)\in\vec{W}_h} \frac{|a(\vec{v}_h,\vec{w}_h) + k(\omega^3_h,\theta^3_h)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)]|}{(\|\vec{w}_h\|^2_{\vec{V}} + \|\theta^3_h\|^2_{0,\Omega})^{1/2}}] + \\
&+ \sup_{(\vec{w}_h,\theta^3_h)\in\vec{W}_h} \frac{|f(\vec{w}_h) - G_h[(\vec{w}_h,\theta^3_h)]|}{(\|\vec{w}_h\|^2_{\vec{V}} + \|\theta^3_h\|^2_{0,\Omega})^{1/2}} \Big) \quad ,
\end{aligned}\right\} \tag{3.1.2}$$

where $\vec{u}$ *(resp.* $(\vec{u}^*_h,\eta^3_h)$*) denotes the solution of the continuous problem (resp. of the discrete problem* 2.3.2*).*

Proof : The assumption (3.1.1) involves the existence and uniqueness of the solution $(\vec{u}^*_h,\eta^3_h)$ of the discrete problem 2.3.2. Then, we can write for any $(\vec{v}_h,\omega^3_h) \in \vec{W}_h$:

$$\beta(\|\vec{u}^*_h-\vec{v}_h\|^2_{\vec{V}} + \|\eta^3_h-\omega^3_h\|^2_{0,\Omega}) \le A_h[(\vec{u}^*_h-\vec{v}_h,\eta^3_h-\omega^3_h),(\vec{u}^*_h-\vec{v}_h,\eta^3_h-\omega^3_h)] \quad ,$$

and by using the fact that $(\vec{u},0)$ is the unique solution of the continuous problem : find $(\vec{u},\eta^3) \in \vec{V}\times L^2(\Omega)$ such that $a(\vec{u},\vec{v}) + k(\eta^3,\omega^3)_{L^2(\Omega)} = f(\vec{v})$; $\forall\vec{v} \in \vec{V}$, $\forall\omega^3 \in L^2(\Omega)$, we obtain :

$$\left.\begin{aligned}&\beta(\|\vec{u}_h^*-\vec{v}_h\|_{\vec{V}}^2+\|\eta_h^3-\omega_h^3\|_{0,\Omega}^2) \le a(\vec{u}-\vec{v}_h,\vec{u}_h^*-\vec{v}_h) + k(-\omega_h^3,\eta_h^3-\omega_h^3)_{L^2(\Omega)} + \\ &+ \{a(\vec{v}_h,\vec{u}_h^*-\vec{v}_h) + k(\omega_h^3,\eta_h^3-\omega_h^3) - A_h[(\vec{v}_h,\omega_h^3),(\vec{u}_h^*-\vec{v}_h,\eta_h^3-\omega_h^3)]\} + \\ &+ \{G_h[(\vec{u}_h^*-\vec{v}_h,\eta_h^3-\omega_h^3)] - f(\vec{u}_h^*-\vec{v}_h)\}\ .\end{aligned}\right\} \quad (3.1.3)$$

Now, the continuity over $(H^1(\Omega))^2 \times H^2(\Omega)$ of the bilinear form $a(.,.)$ involves the existence of a constant M such that : $a(\vec{v},\vec{w}) \le M\|\vec{v}\|_{\vec{V}}\ \|\vec{w}\|_{\vec{V}}$, $\forall\vec{v},\vec{w} \in \vec{V}$, and, thus we get for any $(\vec{v},\omega^3),(\vec{w},\theta^3) \in \vec{V} \times L^2(\Omega)$:

$$a(\vec{v},\vec{w}) + k(\omega^3,\theta^3) \le \sup(M,k)(\|\vec{v}\|_{\vec{V}}^2+\|\omega^3\|_{0,\Omega}^2)^{1/2}(\|\vec{w}\|_{\vec{V}}^2+\|\theta^3\|_{0,\Omega}^2)^{1/2}\ ;$$

therefore, we obtain in the decomposition (3.1.3) :

$$\begin{aligned}(\|\vec{u}_h^*-\vec{v}_h\|_{\vec{V}}^2+\|\eta_h^3-\omega_h^3\|_{0,\Omega}^2)^{1/2} &\le \frac{\sup(M,k)}{\beta}\ (\|\vec{u}-\vec{v}_h\|_{\vec{V}}^2+\|\omega_h^3\|_{0,\Omega}^2)^{1/2} + \\ &+ \frac{1}{\beta}\sup_{(\vec{w}_h,\theta_h^3)\in\vec{W}_h} \frac{|a(\vec{v}_h,\vec{w}_h) + k(\omega_h^3,\theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)]|}{(\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2}} + \\ &+ \frac{1}{\beta}\sup_{(\vec{w}_h,\theta_h^3)\in\vec{W}_h} \frac{|f(\vec{w}_h) - G_h[(\vec{w}_h,\theta_h^3)]|}{(\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2}}\ .\end{aligned}$$

By combining this last inequality with the following triangular inequality :

$$(\|\vec{u}-\vec{u}_h^*\|_{\vec{V}}^2+\|\eta_h^3\|_{0,\Omega}^2)^{1/2} \le (\|\vec{u}-\vec{v}_h\|_{\vec{V}}^2+\|\omega_h^3\|_{0,\Omega}^2)^{1/2} + (\|\vec{v}_h-\vec{u}_h^*\|_{\vec{V}}^2+\|\omega_h^3-\eta_h^3\|_{0,\Omega}^2)^{1/2}\ ,$$

and by taking the minimum with respect to $(\vec{v}_h,\omega_h^3) \in \vec{W}_h$, we deduce the inequality (3.1.2) where $C = \sup(1+\frac{M}{\beta}\ ,\ 1+\frac{k}{\beta}\ ,\ \frac{1}{\beta})$.

□

Let us remark that, in estimate (3.1.2), we find the usual approximation theory term $\inf_{(\vec{v}_h,\omega_h^3)\in\vec{W}_h} (\|\vec{u}-\vec{v}_h\|_{\vec{V}}^2+\|0-\omega_h^3\|_{0,\Omega}^2)^2$, and two *consistency* terms involving the bilinear forms $a(.,.)$ and $A_h[(.,.),(.,.)]$, and the linear forms $f(.)$ and $G_h[(.,.)]$ (compare with (I, (5.1.2)).

3.2. Estimate of $|a(\vec{v}_h,\vec{w}_h) + k(\omega_h^3,\theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)]|$

In order to evaluate the first consistency error, let us consider any pair of functions $(\vec{v}_h,\omega_h^3)$ and $(\vec{w}_h,\theta_h^3) \in \vec{Y}_h$, and let us denote by $(\vec{\vec{v}}_h,\tilde{\omega}_h^3)$ and

$(\vec{\vec{w}}_h, \tilde{\theta}_h^3)$ the functions, that belong to $\vec{\vec{Y}}_h$, respectively associated through the bijection F_h defined in Theorem 2.2.1 ; then, using the correspondences (2.3.6), we have :

$$a(\vec{v}_h, \vec{w}_h) + k(\omega_h^3, \theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h, \omega_h^3), (\vec{w}_h, \theta_h^3)] =$$
$$= \sum_{K \in \mathscr{T}_h} \{[a(\vec{v}_h, \vec{w}_h)_{|K} - \tilde{a}_{Kh}(\vec{\vec{v}}_h, \vec{\vec{w}}_h)] + k[(\omega_h^3, \theta_h^3)_{L^2(K)} - (\tilde{\omega}_h^3, \tilde{\theta}_h^3)_{L^2(K)}]\} .$$

But the estimate of $|a(\vec{v}_h, \vec{w}_h)_{|K} - \tilde{a}_{Kh}(\vec{\vec{v}}_h, \vec{\vec{w}}_h)|$ is obtained through the decomposition (I, (5.2.2)) and the inequality (I, (5.2.3)) ; moreover we have the second estimate :

$$\left.\begin{aligned}
&|(\omega_h^3, \theta_h^3)_{L^2(K)} - (\tilde{\omega}_h^3, \tilde{\theta}_h^3)_{L^2(K)}| = |\int_K (\omega_h^3 \theta_h^3 \sqrt{a} - \tilde{\omega}_h^3 \tilde{\theta}_h^3 \sqrt{a_h})\, d\xi^1 d\xi^2| \\
&\quad \le |\sqrt{a} - \sqrt{a_h}|_{0,\infty,K} \{|\omega_h^3|_{0,K}\, |\theta_h^3|_{0,K} + |\omega_h^3|_{0,K}\, |\theta_h^3 - \tilde{\theta}_h^3|_{0,K} + \\
&\qquad\qquad + |\theta_h^3|_{0,K}\, |\omega_h^3 - \tilde{\omega}_h^3|_{0,K} + |\omega_h^3 - \tilde{\omega}_h^3|_{0,K}\, |\theta_h^3 - \tilde{\theta}_h^3|_{0,K}\} + \\
&\quad + |\sqrt{a}|_{0,\infty,K} \{|\omega_h^3|_{0,K}\, |\theta_h^3 - \tilde{\theta}_h^3|_{0,K} + |\theta_h^3|_{0,K}\, |\omega_h^3 - \tilde{\omega}_h^3|_{0,K} + |\omega_h^3 - \tilde{\omega}_h^3|_{0,K}\, |\theta_h^3 - \tilde{\theta}_h^3|_{0,K}\}
\end{aligned}\right\} \quad (3.2.1)$$

Therefore, the estimate of the first consistency term in (3.1.2) amounts to evaluate the terms :

$$|\gamma_\beta^\alpha(\vec{v}_h) - \tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} , \quad |\overline{\rho}_\beta^\alpha(\vec{v}_h) - \tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} , \quad |\omega_h^3 - \tilde{\omega}_h^3|_{0,K} ,$$

since it was already shown in Part I that $|\sqrt{a} - \sqrt{a_h}|_{0,\infty,K} = 0(h)$ (see (I, (5.2.29))).

Firstly, we obtain the following theorem :

Theorem 3.2.1 : There exists a constant C , *independent of* h , *such that for any* $(\vec{\vec{v}}_h, \tilde{\omega}_h^3) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h, \omega_h^3) \in \vec{Y}_h$ *in correspondence through the bijection* F_h *defined in Theorem 2.2.1, we have*

$$|\gamma_\beta^\alpha(\vec{v}_h) - \tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} \le Ch\{\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{1,K}^2\}^{1/2} . \quad (3.2.2)$$

Proof : This proof is identical with the one of (I, Theorem 5.2.1).

□

Secondly, we prove an extension of (I, Theorem 5.2.2) :

Theorem 3.2.2 : There exists constants $c^{\ell}_{j\alpha\beta}$, $c^{\ell\lambda}_{j\alpha\beta}$, $c^{\epsilon\eta}_{j\alpha\beta}$, $c_{j\alpha\beta}$, *independent of* h , *such that for any* $(\vec{\vec{v}}_h,\tilde{\omega}^3_h) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h,\omega^3_h) \in \vec{Y}_h$ *associated through the bijection* F_h *defined in Theorem 2.2.1, we have*

$$\left.\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} = \\
&\quad \bar{\rho}_{\alpha\beta}(\vec{v}_h)|_{K_j} + a^{\nu\lambda}(\xi)b_{\epsilon\mu}(\xi)(\gamma_{\nu\eta}(\vec{v}_h) + e_{\nu\eta}\sqrt{a(\xi)}[\omega^3_h(\xi) - \frac{1}{2}\,\epsilon^{\tau\sigma}(\xi)v_{h\sigma,\tau}]) \times \\
&\quad \times \sum_{i=1}^{3} A^{\epsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\alpha\beta}] + \\
&\quad + O(h)\sum_{k=1}^{3}(c^{\ell}_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k) + c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k) + c^{\epsilon\eta}_{j\alpha\beta}v_{h3,\epsilon\eta}(\bar{\xi}_k) + c_{j\alpha\beta}\omega^3_h(\bar{\xi}_k)) ,
\end{aligned}\right\}(3.2.3)$$

where $\bar{\xi}_k \in [\xi,\Sigma_k]$, *where we have denoted :*

$$A^{\epsilon\mu}_{\lambda}(\xi) = \frac{1}{2}\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^{\lambda}}(\xi^{\epsilon}_i-\xi^{\epsilon})(\xi^{\eta}_i-\xi^{\eta}) , \tag{3.2.4}$$

and where $p^1_{j,i,i+1}(\lambda)$, $p^1_{j,i,i-1}(\lambda)$ *denote the basis functions for the subtriangle* K_j *of the reduced* H.C.T. *finite element.*

Proof : (in five steps)

By definition of space $\tilde{\tilde{X}}_{h2}$, we get on each subtriangle K_j , $1\leq j\leq 3$, of the triangle K (see (I, (5.2.20))) :

$$\left.\begin{aligned}
&\tilde{v}_{h3,\alpha\beta}(\xi)|_{K_j} = \sum_{i=1}^{3}(p^0_{j,i}(\lambda))_{,\alpha\beta}\,\tilde{v}_{h3}(\Sigma_i) + \\
&\quad + \sum_{i=1}^{3}[(\xi^{\nu}_{i+1}-\xi^{\nu}_{i})(p^1_{j,i,i+1}(\lambda))_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_{i})(p^1_{j,i,i-1}(\lambda))_{,\alpha\beta}]\tilde{v}_{h3,\nu}(\Sigma_i) .
\end{aligned}\right\}(3.2.5)$$

Step 1 : Expression of $\tilde{v}_{h3}(\Sigma_i)$ *as function of degrees of freedom of space* $\vec{Y}_h$

By virtue of compatibility relation (2.2.6), we derive :

$$\tilde{v}_{h3}(\Sigma_i) = d^j_{h3}(\Sigma_i)v_{hj}(\Sigma_i) , \tag{3.2.6}$$

where we have set :

$$d^j_{hk}(\xi) = \vec{a}^j(\xi).\vec{a}_{hk} . \tag{3.2.7}$$

Step 2 : Expression of $\tilde{v}_{h3,\nu}(\Sigma_i)$ *as function of degrees of freedom of space* $\vec{Y}_h$

From the compatibility relation (2.2.7), we have :

$$\left.\begin{aligned}\tilde{v}_{h3,\nu}(\Sigma_i) = \sqrt{\frac{a_h}{a(\Sigma_i)}}\, e_{\lambda\nu} e^{\kappa\mu}[v_{h3,\mu}(\Sigma_i) + b^{\epsilon}_{\mu}(\Sigma_i) v_{h\epsilon}(\Sigma_i)](\vec{a}_{\kappa}(\Sigma_i).\vec{a}^{\lambda}_{h}) +\\ + \sqrt{a_h}\, e_{\lambda\nu}\omega^3_h(\Sigma_i)(\vec{a}_3(\Sigma_i).\vec{a}^{\lambda}_h) \quad .\end{aligned}\right\} \quad (3.2.8)$$

Step 3 : Finite expansion of $\tilde{v}_{h3}(\Sigma_i)$

By noticing that (3.2.6) is identical to (I, (5.2.21)), we record the finite expansion (I, (5.2.37)) :

$$\left.\begin{aligned}\tilde{v}_{h3}(\Sigma_i) = v_{h3}(\Sigma_i) + \{[1 - A^{\sigma\tau}_{\lambda}(\xi)\Gamma^{\lambda}_{\sigma\tau}(\xi)][-A^{\epsilon\eta}_{\mu}(\xi)b_{\epsilon\eta}(\xi)a^{\mu\nu}(\xi) + (\xi^{\epsilon}_i-\xi^{\epsilon})b^{\nu}_{\epsilon}(\xi)] +\\ + [e_{\alpha\beta}e^{\lambda\mu}A^{\epsilon\eta}_{\lambda}(\xi)A^{\gamma\delta}_{\mu}(\xi)\Gamma^{\beta}_{\epsilon\eta}(\xi)b_{\gamma\delta}(\xi) - B^{\epsilon\eta\mu}_{\alpha}(\xi)(b_{\epsilon\eta,\mu}(\xi)+\Gamma^{\kappa}_{\epsilon\eta}(\xi)b_{\kappa\mu}(\xi))]a^{\alpha\nu}(\xi) +\\ + (\xi^{\epsilon}_i-\xi^{\epsilon})[A^{\gamma\delta}_{\alpha}(\xi)b_{\gamma\delta}(\xi)\Gamma^{\nu}_{\epsilon\beta}(\xi)a^{\alpha\beta}(\xi) + A^{\gamma\delta}_{\alpha}(\xi)\Gamma^{\alpha}_{\gamma\delta}(\xi)b^{\nu}_{\epsilon}(\xi) +\\ + \frac{1}{2}(\xi^{\eta}_i-\xi^{\eta})(b^{\nu}_{\epsilon,\eta}(\xi) - \Gamma^{\nu}_{\epsilon\mu}(\xi)b^{\mu}_{\eta}(\xi)]\}v_{h\nu}(\Sigma_i) +\\ + \{-\frac{1}{2a(\xi)}e^{\alpha\beta}e^{\lambda\mu}a_{\beta\mu}(\xi)A^{\epsilon\eta}_{\alpha}(\xi)A^{\gamma\delta}_{\lambda}(\xi)b_{\epsilon\eta}(\xi)b_{\gamma\delta}(\xi) +\\ + (\xi^{\epsilon}_i-\xi^{\epsilon})b^{\alpha}_{\epsilon}(\xi)A^{\gamma\delta}_{\alpha}(\xi)b_{\gamma\delta}(\xi) -\\ - \frac{1}{2}(\xi^{\epsilon}_i-\xi^{\epsilon})(\xi^{\eta}_i-\xi^{\eta})c_{\epsilon\eta}(\xi)\}v_{h3}(\Sigma_i) + 0(h^3)c^j_i v_{hj}(\Sigma_i) \quad ,\end{aligned}\right\} \quad (3.2.9)$$

where we have set :

$$B^{\epsilon\eta\mu}_{\alpha}(\xi) = \frac{1}{6}\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^{\alpha}}(\xi^{\epsilon}_i-\xi^{\epsilon})(\xi^{\eta}_i-\xi^{\eta})(\xi^{\mu}_i-\xi^{\mu}) \quad ,$$

and where the constant c^j_i are independent of h.

Step 4 : Finite expansion of $\tilde{v}_{h3,\nu}(\Sigma_i)$

We will derive the finite expansion of $\tilde{v}_{h3,\nu}(\Sigma_i)$ from the expression (I, (5.2.38)). Indeed, let us rewrite the second compatibility relations (2.2.3) as a linear system, i.e.

$$\frac{1}{\sqrt{a_h}}e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d^{\lambda}_{h\mu}(\Sigma_i) = \frac{1}{\sqrt{a(\Sigma_i)}}e^{\lambda\mu}[v_{h3,\mu}(\Sigma_i) + b^{\epsilon}_{\mu}(\Sigma_i)v_{h\epsilon}(\Sigma_i)] - d^{\lambda}_{h3}(\Sigma_i)\tilde{\omega}^3_h(\Sigma_i) \quad ,$$

$$d^3_{h3}(\Sigma_i)\tilde{\omega}^3_h(\Sigma_i) = \omega^3_h(\Sigma_i) - \frac{1}{\sqrt{a_h}}e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d^3_{h\mu}(\Sigma_i) \quad ,$$

in such a manner that we get by substitution :

$$\left.\begin{aligned}e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)[d^{\lambda}_{h\mu}(\Sigma_i) - \frac{d^{\lambda}_{h3}(\Sigma_i)d^3_{h\mu}(\Sigma_i)}{d^3_{h3}(\Sigma_i)}] =\\ = \sqrt{\frac{a_h}{a(\Sigma_i)}}\,\{e^{\lambda\mu}[v_{h3,\mu}(\Sigma_i) + b^{\epsilon}_{\mu}(\Sigma_i)v_{h\epsilon}(\Sigma_i)] - \sqrt{a(\Sigma_i)}\,\frac{d^{\lambda}_{h3}(\Sigma_i)}{d^3_{h3}(\Sigma_i)}\,\omega^3_h(\Sigma_i)\} \quad .\end{aligned}\right\} \quad (3.2.10)$$

Then, by noticing that we have from (I, (5.2.28) (5.2.32) and (5.2.33)) :

$$\frac{d^{\lambda}_{h3}(\Sigma_i)d^{3}_{h\mu}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)} = 0(h^2) \quad \text{and} \quad \frac{d^{\lambda}_{h3}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)} = d^{\lambda}_{h3}(\Sigma_i) + 0(h^2) \quad ,$$

we obtain in (3.2.10) :

$$\begin{aligned} &e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d^{\lambda}_{h\mu}(\Sigma_i) = \\ &\quad = \sqrt{\frac{a_h}{a(\Sigma_i)}}\,\{e^{\lambda\mu}[v_{h3,\mu}(\Sigma_i) + b^{\epsilon}_{\mu}(\Sigma_i)v_{h\epsilon}(\Sigma_i)] - d^{\lambda}_{h3}(\Sigma_i)\sqrt{a(\Sigma_i)}\omega^{3}_{h}(\Sigma_i)\} + \\ &\quad + 0(h^2)\{c^{\nu}\tilde{v}_{h3,\nu}(\Sigma_i) + c\omega^{3}_{h}(\Sigma_i)\} \ . \end{aligned}$$

Consequently, the terms $\tilde{v}_{h3,\nu}(\Sigma_i)$ are solutions of a linear system similar to the one obtained in (I, Step 2 of the proof of Theorem 5.2.2) in the $0(h^2)$-approximation sense, so that, with the help of the expressions (I, (5.2.22), (5.2.23)), we derive the approximate expression :

$$\left.\begin{aligned} \tilde{v}_{h3,\nu}(\Sigma_i) = &\sqrt{\frac{a_h}{a(\Sigma_i)}}\,\frac{d^{\eta}_{h\nu}(\Sigma_i)}{d_h(\Sigma_i)}\{v_{h3,\eta}(\Sigma_i) + b^{\epsilon}_{\eta}(\Sigma_i)v_{h\epsilon}(\Sigma_i)\} - \\ &- e_{\lambda\eta}\,\frac{d^{\eta}_{h\nu}(\Sigma_i)d^{\lambda}_{h3}(\Sigma_i)}{d_h(\Sigma_i)}\sqrt{a_h}\;\omega^{3}_{h}(\Sigma_i) + \\ &+ 0(h^2)\{c^{\lambda}_{\nu}v_{h\lambda}(\Sigma_i) + c^{3\lambda}_{\nu}v_{h3,\lambda}(\Sigma_i) + c^{3}_{\nu}\omega^{3}_{h}(\Sigma_i)\} \ . \end{aligned}\right\} \quad (3.2.11)$$

Thus, we deduce the following finite expansion (in a similar manner as in (I, Step 5 of the proof of Theorem 5.2.2)) :

$$\left.\begin{aligned} \tilde{v}_{h3,\nu}(\Sigma_i) = &\, v_{h3,\nu}(\Sigma_i) + b^{\mu}_{\nu}(\Sigma_i)v_{h\mu}(\Sigma_i) + \\ &+ [A^{\epsilon\eta}_{\nu}(\xi)\Gamma^{\lambda}_{\epsilon\eta}(\xi) - (\xi^{\epsilon}_{i}-\xi^{\epsilon})\Gamma^{\lambda}_{\nu\epsilon}(\xi)][v_{h3,\lambda}(\Sigma_i) + b^{\mu}_{\lambda}(\Sigma_i)v_{h\mu}(\Sigma_i)] + \\ &+ e_{\gamma\nu}[A^{\epsilon\eta}_{\lambda}(\xi)b_{\epsilon\eta}(\xi)a^{\gamma\lambda}(\xi) - (\xi^{\epsilon}_{i}-\xi^{\epsilon})b^{\gamma}_{\epsilon}(\xi)]\,\sqrt{a(\Sigma_i)}\;\omega^{3}_{h}(\Sigma_i) + \\ &+ 0(h^2)\sum_{k=1}^{3}\{c^{\ell}_{\nu}v_{h\ell}(\Sigma_k) + c^{\ell\lambda}_{\nu}v_{h\ell,\lambda}(\Sigma_k) + c_{\nu}\omega^{3}_{h}(\Sigma_k)\} \ , \end{aligned}\right\} \quad (3.2.12)$$

where the constants c^{ℓ}_{ν} , $c^{\ell\lambda}_{\nu}$, c_{ν} are independent of h.

□

Step 5 : Finite expansion of $\tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)$

By substituting (3.2.9) and (3.2.12) into (3.2.5), we get a similar expression as (I, (5.2.39)) : as it can be noticed in the preceding step, we only have to replace the former term $[-\frac{1}{2}(v_{h1,2}-v_{h2,1})] = [\frac{1}{2}e^{\tau\sigma}v_{h\sigma,\tau}]$, by the latter one : $[\sqrt{a(\Sigma_i)}\ \omega_h^3(\Sigma_i)]$, or simply by $\sqrt{a(\xi)}\ \omega_h^3(\xi)$. Then, by using the arguments that lead to (I, (5.2.51)), we obtain

$$\begin{aligned}
\tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} &= v_{h3,\alpha\beta}(\xi) + \frac{1}{2}\,[b^\lambda_{\alpha,\beta}(\xi) + b^\lambda_{\beta,\alpha}(\xi)]v_{h\lambda}(\xi) + \\
&+ b^\lambda_\alpha(\xi)v_{h\lambda,\beta}(\xi) + b^\lambda_\beta(\xi)v_{h\lambda,\alpha}(\xi) - \Gamma^\lambda_{\alpha\beta}(\xi)[v_{h3,\lambda}(\xi) + b^\nu_\lambda(\xi)v_{h\nu}(\xi)] + \\
&+ \sum_{i=1}^{3} \{(\xi^\epsilon_i-\xi^\epsilon)[R^3_{\lambda 1}(\xi)\Gamma^\nu_{\epsilon\mu}(\xi)a^{\lambda\mu}(\xi) - \frac{1}{2}\,(\xi^\eta_i-\xi^\eta)b^\lambda_\eta(\xi)\Gamma^\nu_{\lambda\epsilon}(\xi)]v_{h\nu}(\xi) + \\
&+ (\xi^\epsilon_i-\xi^\epsilon)[R^3_{\lambda 1}(\xi)b^\lambda_\epsilon(\xi) - \frac{1}{2}\,(\xi^\eta_i-\xi^\eta)b^\lambda_\epsilon(\xi)b_{\lambda\eta}(\xi)]v_{h3}(\xi) - \\
&- (\xi^\epsilon_i-\xi^\epsilon)A^{\lambda\eta}_\mu(\xi)b_{\lambda\eta}(\xi)a^{\nu\mu}(\xi)v_{h\nu,\epsilon}(\xi)\}(p^o_{j,i})_{,\alpha\beta} + \\
&+ \sum_{i=1}^{3} \{(\xi^\nu_{i+1}-\xi^\nu_i)(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^\nu_{i-1}-\xi^\nu_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} \times \\
&\quad \times \{e_{\gamma\nu}[(A^{\epsilon\eta}_\lambda(\xi)b_{\epsilon\eta}(\xi)a^{\gamma\lambda}(\xi) - (\xi^\epsilon_i-\xi^\epsilon)b^\gamma_\epsilon(\xi)]\ \sqrt{a(\xi)}\ \omega_h^3(\xi) + \\
&\quad + (\xi^\epsilon_i-\xi^\epsilon)\ [\ \frac{1}{2}\,(b^\lambda_{\nu,\epsilon}(\xi)-b^\lambda_{\epsilon,\nu}(\xi))v_{h\lambda}(\xi) - b^\lambda_\epsilon(\xi)v_{h\lambda,\nu}(\xi)]\} + \\
&+ O(h)\sum_{i=1}^{3} [c^\ell_{j\alpha\beta}v_{h\ell}(\overline{\xi}_k) + c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\overline{\xi}_k) + c^{\epsilon\eta}_{j\alpha\beta}v_{h3,\epsilon\eta}(\overline{\xi}_k) + c_{j\alpha\beta}\omega^3_h(\overline{\xi}_k)]\ ,
\end{aligned}$$

where we have denoted : $R^3_{\lambda 1}(\xi) = A^{\gamma\delta}_\lambda(\xi)b_{\gamma\delta}(\xi)$, and where the points $\overline{\xi}_k \in [\xi,\Sigma_k]$ can change from one expression to the next. Thus, the technical developments of (I, Steps 6 and 7 of Theorem 5.2.2) give without any difficulty the estimate (3.2.3).

□

Remark 3.2.1 : This last result raises some remarks :

(i) the counterexample of a right circular cylindrical shell (I, lemma 5.2.1) proves that the residual term of the expression (3.2.3), i.e.

$$\begin{aligned}
& a^{\nu\lambda}(\xi)b_{\epsilon\mu}(\xi)\{\gamma_{\nu\eta}(\vec{v}_h) + e_{\nu\eta}\ \sqrt{a(\xi)}\ [\omega^3_h(\xi) - \frac{1}{2}\,\epsilon^{\tau\sigma}(\xi)v_{h\sigma,\tau}]\} \times \\
&\quad \times \sum_{i=1}^{3} A^{\epsilon\mu}_\lambda(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}]\ ,
\end{aligned}$$

does not, in general, reduce to an O(h)-estimate. Indeed, from this study, it is shown that some extraneous bending effects are to be expected, especially in regions where the membrane state of stresses predominates ;

(ii) it appears here that a modification of the compatibility conditions (with respect to the previous method studied in Part I) does not seem able to improve convergence results.

□

The case of "quasi-shallow" shells

In order to pursue the analysis, we introduce some restrictions on the geometry of the middle surface which are consistent with KOITER's formulation of *shallow shell* theory. In this way, we refer to definitions 5.2.1 and 5.2.2 of part I which describe a class of middle surface $\mathscr{S}_\epsilon$ satisfying uniformly on $\bar{\Omega}$:

$$|b_\alpha^\beta| \ , \ |b_{\alpha|\lambda}^\beta| \le \epsilon \quad , \quad \alpha,\beta,\gamma = 1,2 \ , \tag{3.2.13}$$

where $\epsilon>0$ is a "small" parameter. The previous inequality expresses that the normal curvatures and their variation are small.

Now, let us state the following theorem :

Theorem 3.2.3 : There exists positive constants ϵ_0 , h_0 , c_1 , c_2 , *independent of* h , *such that for any* ϵ , $0\le\epsilon\le\epsilon_0$, *and for any* h , $0<h<h_0$, *we have for any shell* $\mathscr{S}_\epsilon$ *satisfying* (3.2.13) *uniformly on* $\bar{\Omega}$:

$$\left.\begin{aligned} &|\bar{\rho}_\beta^\alpha(\vec{v}_h) - \tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} \le \\ &\quad \le (c_1\epsilon+c_2h)(\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{2,K}^2 + \|\omega_h^3\|_{0,K}^2)^{1/2} \ , \end{aligned}\right\} \tag{3.2.14}$$

for any $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h,\omega_h^3) \in \vec{Y}_h$ *associated through the bijection* F_h *defined in Theorem 2.2.1.*

Proof (similar to (I, Theorem 5.2.3)) :

From the estimate (3.2.3), the results of interpolation theory in Sobolev spaces (see CIARLET (1978, Theorem 3.1.2)) lead to the estimate :

$$\left.\begin{aligned} &|\tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h) - \bar{\rho}_\beta^\alpha(\vec{v}_h) - a_h^{\alpha\nu}\Delta_{\nu\beta}(\vec{v}_h,\theta_h^3)|_{0,K} \le \\ &\quad \le c_3h(\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{2,K}^2 + \|\omega_h^3\|_{0,K}^2)^{1/2} \end{aligned}\right\} \tag{3.2.15}$$

where we have denoted :

$$\Delta_{\alpha\beta}(\vec{v}_h,\omega_h^3) = \sum_{i=1}^{3} A_\lambda^{\epsilon\mu}(\Sigma_i)[(\xi_{i+1}^\eta-\xi_i^\eta)(p_{j,i,i+1}^1)_{,\alpha\beta} + (\xi_{i-1}^\eta-\xi_i^\eta)(p_{j,i,i-1}^1)_{,\alpha\beta}] \times$$

$$\times\, a^{\nu\lambda}(\xi)b_{\epsilon\mu}(\xi)\{\gamma_{\nu\eta}(\vec{v}_h) + e_{\nu\eta}\sqrt{a(\xi)}\ [\omega_h^3(\xi) - \tfrac{1}{2}\,\epsilon^{\tau\sigma}(\xi)v_{h\sigma,\tau}]\} \ .$$

Moreover, the constant c_3 is independent of h, and may be taken independent of ϵ , as the parameter ϵ is chosen in the compact set $[0,\epsilon_0]$.

From the expression (3.2.4), it is seen that : $A_\lambda^{\epsilon\mu}(\Sigma_i) = 0(h)$, and thus we derive from the inequalities (3.2.13) :

$$|a^{\nu\lambda}(\xi)b_{\epsilon\mu}(\xi)(\gamma_{\nu\eta}(\vec{v}_h) + e_{\nu\eta}\sqrt{a(\xi)}\ [\omega_h^3(\xi) - \frac{1}{2}\ \epsilon^{\tau\sigma}(\xi)v_{h\sigma,\tau}])|_{0,K} \le$$
$$\le c_4\epsilon(\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{0,K} + \|\omega_h^3\|^2_{0,K})^{1/2} \ ,$$

where c_4 is a positive constant independent of h. In such a manner, we get the estimate :

$$\left.\begin{aligned}&|\Delta_{\alpha\beta}(\vec{v}_h,\omega_h^3)|_{0,K_j} \le\\ &\quad\le c_5\epsilon h^2_{K_j}(\sum_{i=1}^{3}[|(p^1_{j,i,i+1}),_{\alpha\beta}|_{0,\infty,K_j} + |(p^1_{j,i,i-1}),_{\alpha\beta}|_{0,\infty,K_j}]) \times\\ &\qquad\times(\|v_{h1}\|^2_{1,K_j} + \|v_{h2}\|^2_{1,K_j} + \|v_{h3}\|^2_{0,K_j} + \|\omega_h^3\|^2_{0,K_j})^{1/2} \ ,\end{aligned}\right\} \quad (3.2.16)$$

where c_5 is a positive constant, independent of h and ϵ .

Then, by introducing a reference triangle $\hat{K}$ (for example with the vertices $\hat{\Sigma}_1 = (0,0)$, $\hat{\Sigma}_2 = (1,0)$, $\hat{\Sigma}_3 = (0,1)$), and by denoting $\mathscr{F}_{K_j}$ the affine mapping that associates the triangle $\hat{K}$ to the triangle K_j , such that : $\xi = \mathscr{F}_{K_j}(\lambda) \in K_j$ (and, in particular, such that to each basis function $\hat{p}(\lambda)$ defined on $\hat{K}$ we can define the basis function $p_j(\xi) = (\hat{p} \circ \mathscr{F}^{-1}_{K_j})(\xi)$ defined on K_j), we obtain (for $k = i+1$ or $i-1$) :

$$\begin{aligned}|(p^1_{j,i,k}(\lambda)),_{\alpha\beta}|_{0,\infty,K_j} &\le c_6\|\mathscr{F}^{-1}_{K_j}\|^2_{1,\infty,\hat{K}}\ |(\hat{p}_{i,k}(\lambda)),_{\alpha\beta}|_{0,\infty,\hat{K}}\\ &\le c_7 h^{-2}_{K_j}|\hat{p}^1_{i,k}(\lambda)|_{2,\hat{K}}\\ &\le c_8 h^{-2}_{K_j} \ ,\end{aligned}$$

where constants c_6 to c_8 are independent of h and ϵ. Therefore, the last estimate gives in (3.2.16) :

$$|\Delta_{\alpha\beta}(\vec{v}_h,\omega_h^3)|_{0,K} \le c_9\epsilon(\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{0,K} + \|\omega_h^3\|^2_{0,K})^{1/2} \quad (3.2.17)$$

where c_9 is a positive constant independent of h and ϵ.

Finally, by combining estimates (3.2.15) and (3.2.17) with a triangular inequality, we deduce (3.2.14).

□

Thus, we find :

Theorem 3.2.4 : There exists constants $\epsilon_o>0$, $h_o>0$, and $C > 0$, *independent of* h *and* ϵ , *such that for any* ϵ , $0\le\epsilon\le\epsilon_o$, *for any* h , $0<h<h_o$, *and for any pair of functions* $(\vec{\vec{v}}_h,\tilde{\omega}^3_h)$, $(\vec{\vec{w}}_h,\tilde{\theta}^3_h) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h) \in \vec{Y}_h$ *respectively in correspondence through the bijection* F_h *defined in Theorem 2.2.1, we have :*

$$\left.\begin{aligned}&\sum_{K\in\mathcal{T}_h} |a(\vec{v}_h,\vec{w}_h)_{|K} - \tilde{a}_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h)| \le \\ &\qquad \le C(\epsilon+h)(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega})^{1/2}(\|\vec{w}_h\|^2_{\vec{V}}+\|\theta^3_h\|^2_{0,\Omega})^{1/2} \ .\end{aligned}\right\} \qquad (3.2.18)$$

Proof : The proof is identical to the one of (I, Theorem 5.2.4) and relies upon the estimates (3.2.2) and (3.2.14).

□

Next, we get (without any restrictions on the geometry of the shell) the following result :

Theorem 3.2.5 : There exists a constant C , *independent of* h , *such that for any* $(\vec{\vec{v}}_h,\tilde{\omega}^3_h) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h,\omega^3_h) \in \vec{Y}_h$ *in correspondence through the bijection* F_h *defined in Theorem 2.2.1, we have :*

$$|\omega^3_h-\tilde{\omega}^3_h|_{0,K} \le Ch(\|v_{h1}\|^2_{0,K} + \|v_{h2}\|^2_{0,K} + \|v_{h3}\|^2_{1,K} + \|\omega^3_h\|^2_{0,K})^{1/2} \ . \qquad (3.2.19)$$

Proof : From the definition of the space $\tilde{\tilde{X}}_{h1}$, we get for any $\xi \in K$:

$$\tilde{\omega}^3_h(\xi) = \sum_{i=1}^{3} \lambda_i\tilde{\omega}^3_h(\Sigma_i) \quad ,$$

and by virtue of the compatibility relation (2.2.8), we get :

$$\left.\begin{aligned}\tilde{\omega}^3_h(\xi) = &\sum_{i=1}^{3} \lambda_i\{ \frac{1}{\sqrt{a(\Sigma_i)}} e^{\mu\nu}[v_{h3,\nu}(\Sigma_i) + b^\epsilon_\nu(\Sigma_i)v_{h\epsilon}(\Sigma_i)](\vec{a}_\mu(\Sigma_i).\vec{a}^3_h) + \\ &+ \omega^3_h(\Sigma_i)(\vec{a}_3(\Sigma_i).\vec{a}^3_h)\} \ .\end{aligned}\right\} \qquad (3.2.20)$$

Moreover, from (I, (5.2.32), (5.2.33)), we have : $\vec{a}_\mu(\Sigma_i).\vec{a}^3_h = 0(h)$, and $\vec{a}_3(\Sigma_i).\vec{a}^3_h = 1 + 0(h^2)$. Consequently, we find :

$$\tilde{\omega}^3_h(\xi) = \omega^3_h(\xi) + 0(h) \sum_{k=1}^{3}(c^\epsilon v_{h\epsilon}(\Sigma_k) + c^\nu v_{h3,\nu}(\Sigma_k) + c^3\omega^3_h(\Sigma_k)) \quad ,$$

from which the estimate (3.2.19) is derived.

□

From this result, it follows :

Theorem 3.2.6 : There exists a constant C , *independent of* h , *such that for any* $(\vec{\vec{v}}_h,\tilde{\omega}^3_h),(\vec{\vec{w}}_h,\tilde{\theta}^3_h) \in \vec{\vec{Y}}_h$ *and* $(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h) \in \vec{Y}_h$ *respectively associated through the bijection* F_h *defined in Theorem 2.2.1, we have :*

$$\sum_{K\in\mathcal{T}_h} |(\omega^3_h,\theta^3_h)_{L^2(K)} - (\tilde{\omega}^3_h,\tilde{\theta}^3_h)_{L^2(K)}| \le Ch(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega})^{1/2}(\|\vec{w}_h\|^2_{\vec{V}}+\|\theta^3_h\|^2_{0,\Omega})^{1/2} \quad (3.2.21)$$

Proof : It is an immediate consequence of the inequality (3.2.1) and of the estimate (3.2.19).

□

Finally, from theorems 3.2.4 and 3.2.6, we establish the first consistency error estimate :

Theorem 3.2.7 : There exists constants $\epsilon_o>0$, $h_o>0$, *and* $C>0$ *independent of* h *and* ϵ , *such that for any* ϵ , $0\le\epsilon\le\epsilon_o$, *for any* h , $0<h<h_o$, *and for any pair of functions* $(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h) \in \vec{Y}_h$, *we have for any shell* $\mathcal{S}_\epsilon$ *satisfying* (3.2.13) *uniformly on* $\bar{\Omega}$.

$$\left.\begin{aligned} &|a(\vec{v}_h,\vec{w}_h) + k(\omega^3_h,\theta^3_h)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)]| \le \\ &\qquad \le C(h+\epsilon)(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega})^{1/2}(\|\vec{w}_h\|^2_{\vec{V}}+\|\theta^3_h\|^2_{0,\Omega})^{1/2} \ . \end{aligned}\right\} \quad (3.2.22)$$

where $A_h[\cdot,\cdot]$ *is defined by relation* (2.3.7).

Proof : It suffices to combine estimates (3.2.18) and (3.2.21) as it was previously shown. Let us remark that the constant c in (3.2.22) is linearly dependent of k.

□

3.3. Existence and uniqueness of a solution for the discrete problem

The abstract error estimate (3.1.2) relies upon the uniform $\vec{W}_h$-ellipticity of the bilinear form $A_h[(\cdot,\cdot),(\cdot,\cdot)]$, i.e. the inequality (3.1.1) if we equip $\vec{W}_h$ with the norm $(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega})^{1/2}$ for any $(\vec{v}_h,\omega^3_h) \in \vec{W}_h$. Hereunder, we establish this property (3.1.1), and we derive the existence and uniqueness result :

Theorem 3.3.1 : There exists constants $\epsilon_1>0$, $h_1>0$, $k_o>0$ *and* $\beta>0$, *independent of* h *and* ϵ , *such that for any* ϵ , $0\le\epsilon\le\epsilon_1$, *for any* h , $0<h<h_1$, *for any* $k\ge k_o$, *we have for any shell* $\mathcal{S}_\epsilon$ *satisfying* (3.2.13) *uniformly on* $\bar{\Omega}$:

$$\beta(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega}) \le A_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \quad , \quad \forall(\vec{v}_h,\omega^3_h) \in \vec{W}_h \quad . \tag{3.3.1}$$

Then, there exists a unique solution $(\vec{u}^*_h,\eta^3_h) \in \vec{W}_h$ *for the discrete problem* 2.3.2 , *and consequenlty, the discrete problem* 2.3.1 *admits a unique solution* $(\vec{\vec{u}}_h,\tilde{\eta}^3_h)$ *defined through the bijection* F_h *of Theorem 2.2.1* , *i.e.* $(\vec{\vec{u}}_h,\tilde{\eta}^3_h) = F_h^{-1}(\vec{u}^*_h,\eta^3_h)$.

Proof : For any $(\vec{v}_h,\omega^3_h) \in \vec{Y}_h$, we use the decomposition

$$A_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] = a(\vec{v}_h,\vec{v}_h) + k\|\omega^3_h\|^2_{0,\Omega} + \\ + (A_h[\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] - a(\vec{v}_h,\vec{v}_h) - k\|\omega^3_h\|^2_{0,\Omega}) \ ,$$

and by virtue of the uniform $\vec{V}$-ellipticity of the bilinear form $a(.,.)$, i.e. there exists a constant $\alpha>0$, independent of h , such that $a(\vec{v},\vec{v}) \ge \alpha\|\vec{v}\|^2_{\vec{V}}$, $\forall\vec{v} \in \vec{V}$, we get from the estimate (3.2.22) :

$$A_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \ge \alpha\|\vec{v}_h\|^2_{\vec{V}} + k\|\omega^3_h\|^2_{0,\Omega} - C(h+\epsilon)(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega}) \quad .$$

By using similar arguments as in (I, Theorem 5.4.1), we can choose a constant α_o , $0<\alpha_o\le\alpha$, independent of ϵ (by working on compact sets). Therefore, we have :

$$A_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \ge [\min(\alpha_o,k) - C(h+\epsilon)](\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega}) \quad . \tag{3.3.2}$$

The inequality (3.3.1) arises from (3.3.2) by choosing h and ϵ sufficiently small, i.e. $0<h<h_1\le h_o$ and $0<\epsilon<\epsilon_1\le\epsilon_o$, and by taking for k a value sufficiently "large", i.e. $k\ge k_o$. For example, let us fix the values of α_o and k_o, then we would choose : $h_1+\epsilon_1 = \dfrac{\min(\alpha_o,k_o)}{2C}$, and the property (3.3.1) is established for $\beta = \frac{1}{2}\min(\alpha_o,k_o)$.

Thus, we derive the existence and uniqueness of the solution $(\vec{u}^*_h,\eta^3_h)$ for the discrete problem 2.3.2 (as the bilinear form $A_h[(.,.),(.,.)]$ is positive definite over the space $\vec{W}_h$), and, by virtue of Theorem 3.3.1, the unique solution of the discrete problem 2.3.1 is $(\vec{\vec{u}}_h,\tilde{\eta}^3_h) = F_h^{-1}(\vec{u}^*_h,\eta^3_h)$.

□

Remark 3.3.1 : To ensure the numerical stability of the method, it is clear from the previous proof that k should not be taken too small ; at least the choice $k \sim C(h+\epsilon)$ should ensure the existence and uniqueness of the solution

$(\vec{u}^*_h, \eta^3_h)$ (when $\min(\alpha_o, k) = k$), but it would result in a ill-conditionned global stiffness matrix ($k \sim 0$) . Nevertheless, as far as the abstract error estimate depends on the value of k (see theorem 3.1.1), it appears convenient to retain the smallest *admissible* values for k, in order to produce only small perturbations of the strain energy of the shell.

□

3.4. *Estimate of* $|f(\vec{w}_h) - G_h[(\vec{w}_h, \theta^3_h)]|$

We give an estimate of the second term of consistency error which appears in the relation (3.1.2) :

Theorem 3.4.1 : For any $\vec{p} \in (L^2(\Omega))^3$, *there exists a constant* C , *independent of* h , *such that we have for any* $(\vec{w}_h, \theta^3_h) \in \vec{Y}_h$:

$$|f(\vec{w}_h) - F_h[(\vec{w}_h, \theta^3_h)]| \le Ch(\|\vec{w}_h\|^2_{\vec{V}} + \|\theta^3_h\|^2_{0,\Omega})^{1/2} \|\vec{p}\|_{0,\Omega} \ . \tag{3.4.1}$$

Proof : By using the correspondence (2.3.6), we obtain for any $(\vec{w}_h, \theta^3_h) \in \vec{Y}_h$ and $(\vec{\vec{w}}_h, \tilde{\theta}^3_h) \in \vec{\vec{Y}}_h$ associated through the bijection F_h of Theorem 2.2.1 :

$$|f(\vec{w}_h) - G_h[(\vec{w}_h, \theta^3_h)]| \le \sum_{K \in \mathcal{T}_h} |f(\vec{w}_h)_{|K} - \tilde{f}_{Kh}(\vec{\vec{w}}_h)| \ ,$$

where

$$\left.\begin{aligned} |f(\vec{w}_h)_{|K} - \tilde{f}_{Kh}(\vec{\vec{w}}_h)| &\le |\int_K \vec{p}(\vec{w}_h - \vec{\vec{w}}_h)\sqrt{a}\,d\xi^1 d\xi^2| + |\int_K \vec{p}\vec{\vec{w}}_h(\sqrt{a} - \sqrt{a_h})d\xi^1 d\xi^2| + \\ &\qquad + |\int_K \vec{p}(\vec{w} - \vec{\vec{w}}_h)(\sqrt{a} - \sqrt{a_h})d\xi^1 d\xi^2| \\ &\le (|\sqrt{a}|_{0,\infty,K} \, (\sum_{k=1}^{3} \|w_{hk} - \tilde{w}_{hk}\|^2_{0,K})^{1/2} + \\ &\quad + |\sqrt{a} - \sqrt{a_h}|_{0,\infty,K} [\|\vec{w}_h\|_{0,K} + (\sum_{k=1}^{3} \|w_{hk} - \tilde{w}_{hk}\|^2_{0,K})^{1/2}])\|\vec{p}\|_{0,K} \ . \end{aligned}\right\} \tag{3.4.2}$$

From (I, (5.2.15)) and (2.2.6), we get :

$$\tilde{w}_{h\beta}(\xi) = \sum_{i=1}^{3} \lambda_i \tilde{w}_{h\beta}(\Sigma_i) = \sum_{i=1}^{3} \lambda_i d^j_{h\beta} w_{hj}(\Sigma_i) = w_{h\beta}(\xi) + O(h) \sum_{i=1}^{3} c^j w_{hj}(\Sigma_i)$$

and thus,

$$\|w_{h\beta} - \tilde{w}_{h\beta}\|_{0,K} \le Ch(\sum_{k=1}^{3} |w_{hk}|^2_{0,K})^{1/2} \ . \tag{3.4.3}$$

From (I, (5.2.32), (5.2.33)), (2.2.6), (2.2.7), we obtain :

$$\tilde{w}_{h3}(\xi)\Big|_{K_j} = \sum_{i=1}^{3} p^{o}_{j,i}(\lambda)\tilde{w}_{h3}(\Sigma_i) +$$

$$+ \sum_{i=1}^{3} [((\xi^{\nu}_{i+1}-\xi^{\nu}_{i})(p^{1}_{j,i,i+1}(\lambda) + (\xi^{\nu}_{i-1}-\xi^{\nu}_{i})(p^{1}_{j,i,i-1}(\lambda)]\tilde{w}_{h3,\nu}(\Sigma_i)$$

$$= w_{h3}(\xi)\Big|_{K_j} + 0(h) \sum_{i=1}^{3} (c^{k}_{j}w_{hk}(\Sigma_i) + c^{3\mu}_{j}w_{h3,\mu}(\Sigma_i) + c^{3}_{j}\theta^{3}_{h}(\Sigma_i))$$

and thus :

$$\|w_{h3}-\tilde{w}_{h3}\|_{0,K} \le Ch(\|w_{h1}\|^2_{0,K}+\|w_{h2}\|^2_{0,K}+\|w_{h3}\|^2_{1,K}+\|\theta^3_h\|^2_{0,K})^{1/2} \quad . \tag{3.4.4}$$

Now, by combining the estimates (3.4.3), (3.4.4) with $|\sqrt{a}-\sqrt{a_h}|_{0,\infty,K} = 0(h)$, in (3.4.2) , we deduce :

$$|f(\vec{w}_h)_{|K}-\tilde{f}_{Kh}(\vec{\vec{w}}_h)| \le Ch((\|w_{h1}\|^2_{0,K}+\|w_{h2}\|^2_{0,K}+\|w_{h3}\|^2_{1,K}+\|\theta^3_h\|^2_{0,K})^{1/2}\|\vec{p}\|_{0,\Omega} ;$$

by summation over the triangles K of $\mathcal{T}_h$, we derive the estimate (3.4.1).

□

3.5. Pseudo-convergence and error estimates

Henceforth, we are able to complete the analysis in the case of *quasi-shallow* shells (with small parameter ϵ) :

Theorem 3.5.1 : Let $\mathcal{T}_h$ *be a regular family of triangulations of the domain* Ω . *Let* $\vec{W}_h$ *and* $\vec{\vec{W}}_h$ *be the finite element spaces respectively defined in* (2.2.10) *and* (2.2.9). *If the solution* $\vec{u} \in \vec{V}$ *of the continuous problem* (I, 2.2.5) *belongs to the space* $(H^2(\Omega))^2 \times H^3(\Omega)$, *if the load* $\vec{p}$ *belongs to the space* $(L^2(\Omega))^3$, *then there exists constants* $\epsilon_1>0$, $h_1>0$, $0<k_o<k_1$, *and* $C>0$, *independent of* ϵ *and* h , *such that for any* ϵ , $0<\epsilon<\epsilon_1$, *for any* h , $0<h<h_1$, *for any* k , $k_o\le k\le k_1$, *we have for any shell* $\mathcal{S}_\epsilon$ *satisfying* (3.2.13) *uniformly on* $\bar{\Omega}$.

$$\|\vec{u}-\vec{u}^*_h\|_{\vec{V}} + \|\eta^3_h\|_{0,\Omega} \le C(h[(\|u_1\|^2_{2,\Omega}+\|u_2\|^2_{2,\Omega}+\|u_3\|^2_{3,\Omega})^{1/2} + \|\vec{p}\|_{0,\Omega}] + \epsilon\|\vec{u}\|_{\vec{V}}) \quad . \tag{3.5.1}$$

where $(\vec{u}^*_h,\eta^3_h)$ *denotes the unique solution of the discrete problem* 2.3.2.

Proof : Let us denote by $\Pi_{h\alpha}$ the interpolation operator in $X_{h\alpha}$, $\alpha = 1,2$; we have for the triangles of type (1) : $\|u_\beta-\Pi_{h1}u_\beta\|_{1,\Omega} \le Ch|u_\beta|_{2,\Omega}$, and for the reduced H.C.T. triangle (see CIARLET (1978)) : $\|u_3-\Pi_{h2}u_3\|_{2,\Omega} \le Ch\|u_3\|_{3,\Omega}$. Moreover by noticing that $\eta^3 = 0$ on Ω , and thus $\Pi_{h1}\eta^3 = 0$ on Ω , we directly obtain :

$$\left.\begin{aligned} \inf_{(\vec{v}_h,\omega_h^3)\in\vec{W}_h} \left[(\|\vec{u}-\vec{v}_h\|_{\vec{V}}^2+\|\omega_h^3\|_{0,\Omega}^2)^{1/2}\right] &\le \|\vec{u}-\Pi_h\vec{u}\|_{\vec{V}} \\ &\le Ch(|u_1|_{2,\Omega}^2+|u_2|_{2,\Omega}^2+|u_3|_{3,\Omega}^2)^{1/2} , \end{aligned}\right\} \tag{3.5.2}$$

where $\Pi_h\vec{u} = (\Pi_{h1}u_1 \ , \ \Pi_{h1}u_2 \ , \ \Pi_{h2}u_3)$ (indeed, we have $(\Pi_h\vec{u},0) \in \vec{W}_h$) .

Next, we apply the Theorem 3.1.1, under the assumptions of Theorem 3.3.1 (by choosing k in the compact set $[k_o,k_1]$, see remark 3.3.1) ; on the one hand, we derive from Theorem 3.2.7 :

$$\sup_{(\vec{w}_h,\theta_h^3)\in\vec{W}_h} \frac{|a(\Pi_h\vec{u},\vec{w}_h) + k(\Pi_{h1}\eta^3,\theta_h^3) - A_h[(\Pi_h\vec{u},\Pi_{h1}\eta^3),(\vec{w}_h,\theta_h^3)]|}{(\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2}} \le C(h+\epsilon)\|\Pi_h\vec{u}\|_{\vec{V}} \ ; \tag{3.5.3}$$

on the other hand, from Theorem 3.4.1 :

$$\sup_{(\vec{w}_h,\theta_h^3)\in\vec{W}_h} \frac{|f(\vec{w}_h) - G_h[(\vec{w}_h,\theta_h^3)]|}{(\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2}} \le Ch\|\vec{p}\|_{0,\Omega} \ ; \tag{3.5.4}$$

by combining estimates (3.5.2) to (3.5.4) with (3.1.2), and by noticing that $\|\Pi_h\vec{u}\|_{\vec{V}} \le C\|\vec{u}\|_{\vec{V}}$, we obtain (3.5.1).

□

Remark 3.5.1 : Let us recall (see (I, remark 6.1.1)) that error estimate (3.5.1) are composed with two terms : the first one approaches asymptotically zero, while the second one does not ; at best, in shallow shell theories, the geometrical parameter ϵ is closed to 0.

Otherwise, from the estimate $\|\eta_h^3\|_{0,\Omega} = O(h)$, it is infered in (3.2.3) that we obtain :

$$\begin{aligned} \tilde{\rho}_{h\alpha\beta}(\vec{\vec{u}}_h)\Big|_{K_j} &= \bar{\rho}_{\alpha\beta}(\vec{u}_h^*)\Big|_{K_j} + a^{\nu\lambda}b_{\epsilon\mu}[v_{h\nu,\eta} - b_{\nu\eta}v_{h3}] \times \\ &\times \sum_{i=1}^{3} A_\lambda^{\epsilon\mu}(\Sigma_i)[(\xi_{i+1}^\eta-\xi_i^\eta)(p_{j,i,i+1}^1)_{,\alpha\beta} + (\xi_{i-1}^\eta-\xi_i^\eta)(p_{j,i,i-1}^1)_{,\alpha\beta}] + O(h)\ldots \end{aligned}$$

□

4 A TRIANGULAR PLANE FACET ELEMENT METHOD FOR GENERAL SHELLS

Similarly to (I, paragraph 6.3), we derive a new method which is unconditionnally convergent for arbitrary thin shells. For engineering computations, the method described hereunder presents the advantage that it relies only upon the euclidean coordinates of the vertices of the triangular facets, regardless of an explicit knowledge of the geometry of the shell.

Indeed, from the estimates (3.2.2), (3.2.3), (3.2.19), and

$$\frac{1}{2}\, e^{\tau\sigma} v_{h\sigma,\tau} = \frac{1}{2}\, e^{\tau\sigma} \tilde{v}_{h\sigma,\tau} + O(h) \ldots ,$$

it follows that we have : $|\tilde{\tilde{\rho}}_{h\alpha\beta}(\vec{\tilde{v}}_h, \tilde{\omega}^3_h) - \bar{\rho}_{\alpha\beta}(\vec{v}_h)|_{0,K} = O(h) \ldots$, where we have introduced :

$$\tilde{\tilde{\rho}}_{h\alpha\beta}(\vec{\tilde{v}}_h, \tilde{\omega}^3_h)|_{K_j} = \tilde{v}_{h3,\alpha\beta|K_j} -$$
$$- \sum_{i=1}^{3} A^{\epsilon\mu}_{\lambda}(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta} + (\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \times$$
$$\times\, a_h^{\nu\lambda} b_{h\epsilon\mu}\{\tilde{v}_{h\nu,\eta} + e_{\nu\eta}\sqrt{a_h}\,\tilde{\omega}^3_h\} ,$$

where $b_{h\epsilon\mu}$ is an $O(h)$-approximation of $b_{\epsilon\mu}(\xi)$ (we give an example in section 4.1). Therefore, we are able to find a new discrete method, convergent for arbitrary thin shells, as soon as we can state an approximation of the type : $|b_{h\epsilon\mu} - b_{\epsilon\mu}|_{0,\infty,K} = O(h)$.

4.1. Approximation of the curvatures

In this section, we consider the approximation of the functions $b_{\epsilon\mu}(\xi)$ based on the *isoparametric interpolation of type (2)* ; (in this way, we obtain the estimate $|b_{\epsilon\mu} - b_{h\epsilon\mu}|_{0,\infty,K}$) $O(h)$.

Remark 4.1.1 : Initially, we have considered an approximation of the curvatures based on a (generalized) *six nodes finite differences scheme*, consisting in the formula :

$$b_{h\alpha\beta}(\Sigma) = \left(\frac{1}{n(\Sigma)} \sum_{i=1}^{n(\Sigma)} \vec{a}_{h3|K_i}\right)\left(\sum_{j=0}^{5} c^j_{h\alpha\beta}(\Sigma)\vec{\phi}(\Sigma_j)\right)$$

where $n(\Sigma)$ is the number of triangles K_i that admit the point Σ as a common vertex, and where the coefficients $c^j_{h\alpha\beta}(\Sigma)$ are the solutions of the linear system

$$\sum_{j=0}^{5} c^j_{h\alpha\beta} = 0 \quad ,$$

$$\sum_{j=0}^{5} (\xi^\epsilon_o - \xi^\epsilon_j) c^j_{h\alpha\beta} = 0 \quad , \quad \epsilon = 1,2 \quad ,$$

$$\sum_{j=0}^{5} (\xi^\epsilon_o - \xi^\epsilon_j)(\xi^\eta_o - \xi^\eta_j) c^j_{h\alpha\beta} = (\delta^\epsilon_\alpha \delta^\eta_\beta + \delta^\eta_\alpha \delta^\epsilon_\beta) \quad , \quad \epsilon,\eta = 1,2 \quad ,$$

where we have set $\Sigma = (\xi_o^1, \xi_o^2)$. If the rank of the linear system is 6, we can show an estimate like :

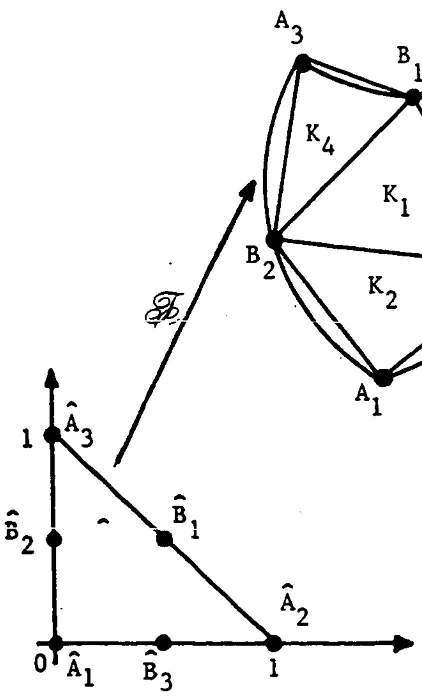
$$|b_{h\alpha\beta}(\Sigma) - b_{\alpha\beta}(\Sigma)| \leq Ch\|\vec{\phi}\|_{3,\infty,\Omega} .$$

But, as far as computation work is concerned, this method appears to be fastidious ; from this point of view, one would prefer the following approach.

□

The "isoparametric interpolation" approach

This more convenient approach give an O(h)-approximation of the curvatures $b_{\alpha\beta}$ on each triangle of $\mathcal{T}_h$. Let us consider a second grid, constituted from the triangulation $\mathcal{T}_h$ by patching four triangles K_i , $1\leq i\leq 4$: a central triangle K_1 with vertices B_j , $1\leq j\leq 3$, and the three adjacent triangles K_2 , K_3 , K_4 , which share with K_1 a common edge (see figure below). Let us denote by A_j , $1\leq j\leq 3$, the three remaining vertices of the patch (respectively attached to the triangle K_{j+1}) . Thus, it is possible to define an *isoparametric finite element* $(\mathcal{K}, \mathcal{P}, \mathcal{S})$ *of type (2)* (see CIARLET (1978, Theorem 4.3.1)) : given a Lagrange finite element $(\hat{\mathcal{K}}, \hat{\mathcal{P}}, \hat{\mathcal{S}})$ in $\mathbb{R}^2$, there exists a one-to-one mapping $\mathcal{F}$ such that :



$$\left.\begin{aligned}
&\mathcal{F} : \hat{x} \in \hat{\mathcal{K}} \to (\mathcal{F}_k(\hat{x}) - \xi_k)_{k=1,2} \in \mathbb{R}^2 , \\
&\mathcal{F}_k \in P_2(\hat{\mathcal{K}}) , k = 1,2 , \mathcal{K} = \mathcal{F}(\hat{\mathcal{K}}) \\
&\mathcal{P} = \{p : \mathcal{K} \to \mathbb{R}^2 , p = \hat{p} \circ \mathcal{F}^{-1} , \hat{p} \in P_2(\hat{\mathcal{K}})\} , \\
&\mathcal{S} = \{p(\mathcal{F}(\hat{A}_i)) , p(\mathcal{F}(\hat{B}_i)) , 1\leq i\leq 3 , p \in \mathcal{P}\}
\end{aligned}\right\} (4.1.$$

The mapping is uniquely determined by the data : $\mathcal{F}(\hat{A}_i) = A_i$, $\mathcal{F}(\hat{B}_i) = B_i$, $1\leq i\leq 3$.

Fig. 4.1. :
The reference Lagrange finite element $(\hat{\mathcal{K}}, P_2(\hat{\mathcal{K}}), \hat{\mathcal{S}})$ *and its isoparametrically equivalent finite element* $(\mathcal{K}, \mathcal{P}, \mathcal{S})$

For instance, in the case of an isoparametric triangle of type (2), the mapping is given by :

$$\left.\begin{aligned}\mathscr{F}: \hat{x} \in \hat{\mathscr{K}} \rightarrow \mathscr{F}(\hat{x}) = \\ = \sum_{i=1}^{3} \lambda_i(\hat{x})(2\lambda_i(\hat{x})-1)A_i + \sum_{i=1}^{3} 4\lambda_{i+1}(\hat{x})\lambda_{i+2}(\hat{x})B_i \; ;\end{aligned}\right\} \quad (4.1.2)$$

this mapping being invertible as soon as the points B_i are "close" to $\frac{1}{2}(A_{i+1}+A_{i+2})$ (see CIARLET (1978, Theorem 4.3.3)), provided that h is small enough.

Next, we have :

Lemma 4.1.1 : Let $\mathscr{T}_h$ *be a regular triangulation of the domain* $\bar{\Omega}$ *. Then, for any triangle* K *of* $\mathscr{T}_h$ *, with vertices* Σ_j $1 \le j \le 3$ *, there exists a patch of four triangles* K_i *of* $\mathscr{T}_h$, $1 \le i \le 4$ *, containing* K *, and a set of six vertices* $\{A_j, B_j, 1 \le j \le 3\}$ *, containing* Σ_j , $1 \le j \le 3$ *, such that, with definitions* (4.1.1) *and* (4.1.2) *, we get :*

$$\left.\begin{aligned}b_{h\alpha\beta|K} = \vec{a}_{h3|K} \cdot \Big\{ \sum_{i=1}^{3} [\lambda_i(\hat{x})(2\lambda_i(\hat{x}) - 1]_{,\alpha\beta}\vec{\phi}(A_i) + \\ + \sum_{i=1}^{3} [4\lambda_{i+1}(\hat{x})\lambda_{i+2}(\hat{x})]_{,\alpha\beta}\vec{\phi}(B_i) \Big\}\end{aligned}\right\} \quad (4.1.3)$$

and, for h *sufficiently small, we obtain :*

$$|b_{\alpha\beta} - b_{h\alpha\beta}|_{0,\infty,K} \le Ch\|\vec{\phi}\|_{3,\infty,\Omega} \quad (4.1.4)$$

where C *is a positive constant, independent of* h .

Proof : On the one hand, we get from (I, (5.2.30), (5.2.31)) :

$$|\vec{a}_3(\Sigma) - \vec{a}_{h3}|_{0,\infty,K} \le C_1 h\|\vec{\phi}\|_{2,\infty,\Omega} \quad . \quad (4.1.5)$$

On the other hand, for any regular triangulation $\mathscr{T}_h$, such that we have :

$$|B_i - \frac{1}{2}(A_{i+1}+A_{i+2})| = O(h^2) \quad , \; 1 \le i \le 3 \quad , \quad (4.1.6)$$

(therefore, we restrict ourselves to the case of a *regular isoparametric family of triangles* $\mathscr{K}$ *of type (2)*, with respect to CIARLET's definition (1978, (4.3.27)), it can be shown that (see CIARLET (1978, Theorem 4.3.4)), for h sufficiently small :

$$|\vec{\phi} - \Pi_{\mathscr{K}}\vec{\phi}|_{2,\infty,\mathscr{K}} \le C_2(2h)\|\vec{\phi}\|_{3,\infty,\Omega} \quad , \quad (4.1.7)$$

where $\Pi_{\mathcal{K}}$ denotes the $\mathcal{P}$-interpolation operator, i.e.

$$\Pi_{\mathcal{K}}\vec{\phi}_{,\alpha\beta} = \sum_{i=1}^{3}[\lambda_i(\hat{x})(2\lambda_i(\hat{x}) - 1)]_{,\alpha\beta}\vec{\phi}(A_i) + \sum_{i=1}^{3}[4\lambda_{i+1}(\hat{x})\lambda_{i+2}(\hat{x})]_{,\alpha\beta}\vec{\phi}(B_i) ,$$

which is constant over $\mathcal{K}$ and thus for any triangle K_i , $1\le i\le 4$, that belongs to $\mathcal{K}$. Thus, from the estimates (4.1.5) and (4.1.7), we derive (4.1.4), by noticing that we have :

$$|b_{\alpha\beta}-b_{h\alpha\beta}|_{0,\infty,K} \le$$

$$\le |\vec{a}_3\cdot(\vec{\phi}_{,\alpha\beta}-\Pi_{\mathcal{K}}\vec{\phi}_{,\alpha\beta})_{|K}| + |(\vec{a}_3-\vec{a}_{h3})_{|K}\cdot\vec{\phi}_{,\alpha\beta|K}| + |(\vec{a}_3-\vec{a}_{h3})_{|K}\cdot(\vec{\phi}_{,\alpha\beta}-\Pi_{\mathcal{K}}\vec{\phi}_{,\alpha\beta})| .$$

□

Remark 4.1.2 : The estimate (4.1.7) can be established under the weaker assumption : $|B_i - \frac{1}{2}(A_{i+1}+A_{i+2})| = O(h^{1+\epsilon})$, for any ϵ , $0<\epsilon<1$, instead of (4.1.6). Indeed, following CIARLET's proof (1978, Theorem 4.3.3), this is sufficient to state that the mapping $\mathcal{F}$ is locally invertible, and estimates remain unchanged. Otherwise, from the regularity assumption of the triangulation $\mathcal{C}_h$, it is conjectured that there exists such an ϵ , $0<\epsilon<1$, for any patch of four triangles of $\mathcal{C}_h$, so that the lemma 4.1.1 is available for any patch in $\mathcal{C}_h$.

□

4.2. A new discrete method with numerical integration

Similarly to the presentations of sections 4.3 and 6.3 of part I, and 2.3 of this part, and taking into the effect of numerical integration, let us introduce the new discrete problem :

Problem 4.2.1 : Find $(\vec{\vec{u}}_h^*,\tilde{\eta}_h^{*3}) \in \vec{\vec{W}}_h$ *such that*

$$\tilde{A}_h^*[(\vec{\vec{u}}_h^*,\tilde{\eta}_h^{*3}),(\vec{\vec{v}}_h,\tilde{\omega}_h^3)] = \tilde{G}_h^*[(\vec{\vec{v}}_h,\tilde{\omega}_h^3)] \quad , \quad \forall(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{W}}_h \quad ,$$

where we have defined a new approximate bilinear form :

$$\tilde{A}_h^*[(\vec{\vec{v}}_h,\tilde{\omega}_h^3),(\vec{\vec{w}}_h,\tilde{\theta}_h^3)] = \sum_{K\in\mathcal{C}_h}[\tilde{a}_{Kh}^*(\vec{\vec{v}}_h,\vec{\vec{w}}_h) + k\tilde{b}_{Kh}^*(\tilde{\omega}_h^3,\tilde{\theta}_h^3)] \quad , \qquad (4.2.1)$$

with :

$$\left.\begin{aligned}&\tilde{a}^*_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h) = \\ &= \sqrt{a_h}\ \Big(\sum_{\ell_1=1}^{L_1}\omega_{\ell_1,K}\{\frac{Ee}{1-\nu^2}\ [(1-\nu)\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{v}}_h)\tilde{\gamma}^{\ \beta}_{h\alpha}(\vec{\vec{w}}_h) + \nu\tilde{\gamma}^{\ \alpha}_{h\alpha}(\vec{\vec{v}}_h)\tilde{\gamma}^{\ \beta}_{h\beta}(\vec{\vec{w}}_h)]\}(b_{\ell_1,K}) + \\ &+\sum_{i=1}^{3}\ \sum_{\ell_2=1}^{L_2}\omega_{\ell_2,K_i}\{\frac{Ee^3}{12(1-\nu^2)}\ [(1-\nu)\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\vec{v}}_h)\tilde{\rho}^{*\beta}_{h\alpha}(\vec{\vec{w}}_h) + \nu\tilde{\rho}^{*\alpha}_{h\alpha}(\vec{\vec{v}}_h)\tilde{\rho}^{*\beta}_{h\beta}(\vec{\vec{w}}_h)]\}(b_{\ell_2,K_i})\Big)\end{aligned}\right\} \quad (4.2.2)$$

with the following expression of the modified change of curvature tensor :

$$\left.\begin{aligned}&\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\vec{v}}_h) = a_h^{\alpha\nu}\tilde{\rho}^*_{h\nu\beta}(\vec{\vec{v}}_h)\ , \\ &\tilde{\rho}^*_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} = \tilde{v}_{h3,\alpha\beta}|_{K_j} - \sum_{i=1}^{3} A^{\epsilon\mu}_{\lambda}(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta} + \\ &\qquad + (\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,j-1})_{,\alpha\beta}]b_{h\epsilon\mu}a_h^{\nu\lambda}\tilde{v}_{h\nu,\eta}\ .\end{aligned}\right\} \quad (4.2.3)$$

The expression of $b_{h\epsilon\mu}$ *is given by* (4.1.3) *; Moreover :*

$$\tilde{b}^*_{Kh}(\tilde{\omega}^3_h,\tilde{\theta}^3_h) = \sqrt{a_h}\ \sum_{\ell_3=1}^{L_3}\omega_{\ell_3,K}\{\tilde{\omega}^3_h\tilde{\theta}^3_h\}(b_{\ell_3,K})\ , \quad (4.2.4)$$

and the new approximate linear form is defined by :

$$\left.\begin{aligned}\tilde{G}^*_h[(\vec{\vec{v}}_h,\tilde{\omega}^3_h)] = \sum_{K\in\mathcal{T}_h}\sqrt{a_h}\ \Big(&\sum_{\ell_1=1}^{L_1}\omega_{\ell_1,K}\{(\vec{p}.\vec{a}^\alpha_h).\tilde{v}_{h\alpha}\}(b_{\ell_1,K}) + \\ &+\sum_{i=1}^{3}\ \sum_{\ell_2=1}^{L_2}\omega_{\ell_2,K_i}\{(\vec{p}.\vec{a}^3_h).\tilde{v}_{h3}\}(b_{\ell_2,K_i})\Big)\ .\end{aligned}\right\} \quad (4.2.5)$$

$\square$

In the expressions (4.2.2), (4.2.4), (4.2.5), we have used three numerical quadrature schemes, namely $\int_K\phi(x)dx \sim \sum_{\ell=1}^{L}\omega_{\ell,K}\phi(b_{\ell,K})$, with $\omega_{\ell,K} = \det(B_K)\hat{\omega}_\ell$ and $b_{\ell,K} = F_K(\hat{b}_\ell)$, $1\le\ell\le L$, where F_K is an invertible affine mapping which associates a reference triangle $\hat{K}$ to the triangle K , i.e. $F_K : \hat{x}\in\hat{K}\to F_K(\hat{x}) = B_K\hat{x} + b_K \in K$, $B_K\in\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$, $b_K\in\mathbb{R}^2$. Subsequently, we will denote by $E_i(.)$, $1\le i\le 3$, the error functional associated with each numerical quadrature scheme.

Now, using the bijection F_h defined in Theorem 2.2.1, it is possible to associate to problem 4.2.1, an equivalent discrete problem formulated on the continuous middle surface, i.e.

Problem 4.2.2 : Find $(\vec{u}_h^{**}, \eta_h^{*3}) \in \vec{W}_h$ *such that*

$$A_h^*[(\vec{u}_h^{**}, \eta_h^{*3}), (\vec{v}_h, \omega_h^3)] = G_h^*[(\vec{v}_h, \omega_h^3)] \quad , \quad \forall (\vec{v}_h, \omega_h^3) \in \vec{W}_h \quad ,$$

with the following correspondences :

$$\left.\begin{aligned} &A_h^*[(\vec{v}_h, \omega_h^3), (\vec{w}_h, \theta_h^3)] = \tilde{A}_h^*[(\vec{\vec{v}}_h, \tilde{\omega}_h^3), (\vec{\vec{w}}_h, \tilde{\theta}_h^3)] \quad , \\ &G_h^*[(\vec{v}_h, \omega_h^3)] = \tilde{G}_h^*[(\vec{\vec{v}}_h, \tilde{\omega}_h^3)] \quad , \end{aligned}\right\} \qquad (4.2.6)$$

for any $(\vec{v}_h, \omega_h^3) = F_h(\vec{\vec{v}}_h, \tilde{\omega}_h^3)$ *, and any* $(\vec{w}_h, \theta_h^3) = F_h(\vec{\vec{w}}_h, \tilde{\theta}_h^3)$.

□

4.3. Convergence and error estimates

From the study developped in paragraph 3, we establish the convergence of the discrete method described in section 4.2, for general shells :

Theorem 4.3.1 : Let $\mathcal{T}_h$ *be a regular family of triangulations of the domain* $\bar{\Omega}$ *. Let* $\vec{W}_h$ *and* $\vec{\vec{W}}_h$ *be the finite element spaces respectively defined in (2.2.10) and (2.2.9). Assume that the numerical integration schemes in (4.2.2), (4.2.4), (4.2.5), satisfy :*

$$\left.\begin{aligned} &\forall \hat{\phi} \in P_o(\hat{K}) \quad , \quad \hat{E}_1(\hat{\phi}) = 0 \text{ (for membrane terms)} \quad , \\ &\forall \hat{\phi} \in P_2(\hat{K}) \quad , \quad \hat{E}_2(\hat{\phi}) = 0 \text{ (for bending terms)} \quad , \\ &\forall \hat{\phi} \in P_o(\hat{K}) \quad , \quad \hat{E}_3(\hat{\phi}) = 0 \text{ (for drilling rotations)} \quad . \end{aligned}\right\} \qquad (4.3.1)$$

Then, if the solution $\vec{u} \in \vec{V}$ *of the continuous problem 2.1.1 belongs to the space* $(H^2(\Omega))^2 \times H^3(\Omega)$ *, if the loads* $\vec{p}$ *belongs to the space* $(W^{1,q}(\Omega))^3$ *,* $q \in \mathbb{R}$ *,* $q>2$ *, there exists constants* $0<k_o<k_1$ *,* $h_1>0$ *, such that for any* k *,* $k_o \le k \le k_1$ *, for any* $h<h_1$ *,*

(i) *the discrete problem (4.2.2) (respectively 4.2.1) has one and only one solution* $(\vec{u}_h^{**}, \eta_h^{*3}) \in \vec{W}_h$ *(resp.* $(\vec{\vec{u}}_h^{*}, \tilde{\eta}_h^{*3}) \in \vec{\vec{W}}_h$*) ;*

(ii) *there exists a positive constant* C *, independent of* h *, such that :*

$$(\|\vec{u}-\vec{u}_h^{**}\|_{\vec{V}}^2 + \|\eta_h^{*3}\|_{0,\Omega}^2)^{1/2} \le Ch\{(\|u_1\|_{2,\Omega}^2 + \|u_2\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2} + \|\vec{p}\|_{1,q,\Omega}\} \quad . \qquad (4.3.2)$$

Proof : This proof follows exactly the lines of the proof of theorem 3.5.1. The interpolation error estimate (3.5.3) is still available here. Next, by noticing that we have :

BERGAN, P.G. ; FELIPPA, C.A. (1985) : A triangular membrane element with rotational degrees of freedom, Comp. Meth. Appl. Mech. Engng., 50, 25-69.

BERNADOU, M. (1980) : Convergence of conforming finite element methods for general shell problems, Int. J. Engng. Sc., 18, 249-276.

BERNADOU, M. ; BOISSERIE, J.M. (1982) : The Finite Element Method in Thin Shell Theory : Application to Arch Dam Simulations, Birkhaüser, Boston.

BERNADOU, M. ; CIARLET, P.G. (1976) : Sur l'ellipticité du modèle linéaire de W.T. Koiter, "Computing Methods in Applied Sciences and Engineering". Lecture Notes in Economic and Mathematical Systems, Vol. 134, 89-136, Springer-Verlag, Berlin.

BERNADOU, M. ; DUCATEL, Y. ; TROUVÉ, P. : Approximation of a circular cylindrical shell by Clough-Johnson flat plate finite elements (to appear in Numer. Math.).

BERNADOU, M. ; DUCATEL, Y. ; TROUVÉ, P. : Approximation of general shell problems by flat plate elements, Part 1, (submitted to Comp. Mech.).

BERNADOU, M. ; ODEN, J.T. (1981) : An existence theorem for a class of nonlinear shallow shell problems, J. Math. Pures et Appl., 60, 285-308.

BERNADOU, M. ; TROUVÉ, P. (to appear) : Approximation of general shell problems by triangular curved facet elements, Part 3 : Extension to triangular curved facet elements.

CARR, A.J. (1967) : A refined finite element analysis of thin shell structures including dynamic loads, Report SESM 67-9, University of California, Berkeley.

CIARLET, P.G. (1976) : Conforming finite element method for the shell problems, in the Mathematics of Finite Elements and Applications II, J.R. Whiteman Ed. Academic Press, London, 105-123.

CIARLET, P.G. (1978) : The Finite Element Method for Elliptic Problems, North-Holland.

CLOUGH, R.W. ; JOHNSON, C.P. (1968) : A finite element approximation for the analysis of thin shells, Int. J. Solids Structures, 4, 43-60.

CLOUGH, R.W. ; JOHNSON, C.P. (1970) : Finite element analysis of arbitrary thin shells, Proceeding ACI Symposium on concrete thin shells, 333-363, New York.

GREEN, A.E. ; ZERNA, W. (1968) : Theoretical Elasticity, Oxford University Press, 2nd Edition.

$$\left.\begin{aligned}\tilde{\rho}^{*}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h)|_{K_j} + \sum_{i=1}^{3} A^{\epsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\alpha\beta} + \\ + (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\alpha\beta}]b_{\epsilon\mu}a^{\nu\lambda}e_{\nu\eta}\sqrt{a}\,\omega^3_h + \\ + O(h)\sum_{k=1}^{3}\{c^{\ell}_{j\alpha\beta}v_{h\ell}(\xi_k) + c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\xi_k) + c^{\epsilon\eta}_{j\alpha\beta}v_{h3,\epsilon\eta}(\xi_k) + c_{j\alpha\beta}\omega^3_h(\xi_k)\}\end{aligned}\right\}(4.3.3)$$

(by virtue of the theorem 3.2.2, the lemma 4.1.1 and the definition (4.2.3)), we derive the estimate (in a similar manner as in theorem 3.2.3)

$$|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\vec{v}}_h)|_{0,K} \le C_1h_1(\|v_{h1}\|^2_{1,K}+\|v_{h2}\|^2_{1,K}+\|v_{h3}\|^2_{2,K})^{1/2} + C_2(h+\epsilon)\|\omega_{h3}\|_{0,K} \quad ,(4.3.4)$$

where C_1 and C_2 are two constants, independent of h, and $\epsilon = \sup_{\xi\in K}|b_{\epsilon\mu}(\xi)|$.

Otherwise, as far as the effect of numerical integration is considered, we obtain similar results as in theorems 5.2.5 to 5.2.7 of part I, i.e. : $\|\vec{\vec{v}}_h\|_h \le C\|\vec{v}_h\|$, $\|\tilde{\omega}^3_h\|_{0,K} \le C\|\omega^3_h\|_{0,K}$, $|E_{K_M}(\vec{\vec{v}}_h,\vec{\vec{w}}_h)| = O(h)$, $|E^*_{K_{iB}}(\vec{\vec{v}}_h,\vec{\vec{w}}_h)| = O(h)$ and $|E_{K_R}(\tilde{\omega}^3_h,\tilde{\theta}^3_h)| = O(h)$, with obvious notations. Henceforth, as soon as $\epsilon_\Omega = \sup_{\xi\in\bar{\Omega}}|b_{\epsilon\mu}(\xi)| < +\infty$, it can be proved that :

$$\left.\begin{aligned}|a(\vec{v}_h,\vec{w}_h) + k(\omega^3_h,\theta^3_h)_{L^2(\Omega)} - A^*_h[(\vec{v}_h,\omega^3_h),(\vec{w}_h,\theta^3_h)]| \le \\ \le C_3h(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega})^{1/2}(\|\vec{w}_h\|^2_{\vec{V}}+\|\theta^3_h\|^2_{0,\Omega})^{1/2} + C_4\epsilon_\Omega\|\omega^3_h\|_{0,\Omega}\,\|\theta^3_h\|_{0,\Omega} \,,\end{aligned}\right\}\quad(4.3.5)$$

where C_3 and C_4 denotes two positive constants, independent of h. Then, by using the uniform $\vec{V}$-ellipticity of the bilinear form a(.,.) (see the proof of theorem 3.3.1) we have on the one hand :

$$A^*_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \ge [\min(\alpha,\tfrac{k}{2})-C_3h].(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega}) + [\tfrac{k}{2} - C_4\epsilon_\Omega]\|\omega^3_h\|^2_{0,\Omega} \,,$$

and thus, by taking $k_o = 2C_4\epsilon_\Omega$ and $h_1 = \min(\alpha,\frac{k_o}{2})/2C_3$, we obtain for any $k\ge k_o$ and for any $h<h_1$:

$$\beta^*(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega^3_h\|^2_{0,\Omega}) \le A^*_h[(\vec{v}_h,\omega^3_h),(\vec{v}_h,\omega^3_h)] \quad , \quad \forall(\vec{v}_h,\omega^3_h) \in \vec{W}_h \quad , \qquad (4.3.6)$$

with $\beta^* = \frac{1}{2}\min(\alpha,\frac{k_o}{2})$. Firstly, the inequality (4.3.6) involves the existence and uniqueness of the solution $(\vec{u}^{**}_h,\eta^{*3}_h) \in \vec{W}_h$ of the discrete problem 4.2.2, and, through the bijection F_h defined in theorem 2.2.1, the existence and uniqueness of the solution $(\vec{\vec{u}}^{*}_h,\tilde{\eta}^{*3}_h) \in \vec{\vec{W}}_h$ of the discrete problem 4.2.1. Secondly, it allows us to obtain an abstract error estimate of the type (3.1.2). On the other hand, the estimate (4.3.5) leads to the first consistency

IDELHSON, S. (1981) : On the use of deep, shallow or flat shell finite elements for the analysis of thin shell structures, Comp. Meth. Appl. Mech. Engng., 26, 321-330.

KIKUCHI, F. (1984) : Error analysis of flat plate element approximation of circular cylindrical shells, Theoretical and Applied Mechanics, 32, 469-484.

KOITER, W.T. (1970) : On the foundations of the linear theory of thin elastic shells, Proc. Kon. Ned. Akad. Wetensch., B73, 169-195.

MORLEY, L.S.D. (1984) : A facet-like shell theory, Int. J. Engng. Sci., 22, 1315-1327.

OLSON, M.D. ; BEARDEN, T.W. (1979) : A simple flat triangular shell element revisited, Int. J. Numer. Meth. Engng., 14, 51-68.

TROUVÉ, P. (to appear) : Approximations nonconformes de problèmes généraux de coques minces par éléments finis du type "Morley", Rapport de Recherche INRIA.

ZIENKIEWICZ, O.C. (1977) : The Finite Element Method in Engineering Science, 3rd Edition, Mc Graw-Hill.

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